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Mathematical Reviews

Vol. 20, No. 4 APRIL, 1959 Reviews 2268-3061

LOGIC AND FOUNDATIONS

See also 2979.

2268:

✶Wittenberg, Alexander Israel. Vom Denken in Begriffen: Mathematik als Experiment des reinen Denkens. Mit einem Geleitwort von P. Bernays. Wissenschaft und Kultur, Bd. 12. Birkhäuser Verlag, Basel-Stuttgart, 1957. 360 pp. DM 25.00.

Dieses, als Doktordissertation verfasste Buch verdient eine eingehende Besprechung.

Die Voraussetzung des Autors ist, "dass für unser Erkennen nicht primär eine Wirklichkeit, sondern nur erst eine Daseinssituation als Ausgangspunkt des Forschens gegeben ist" (p. 15, Bernays). Die Sprache spielt dabei nicht die Rolle eines treuen Reproduzenten von ideenmässig jenseitig und absolut gegebenen Entitäten sondern ist ein aktiver, sich mit der Zeit und mit dem wissenschaftlichen Gebiet wandelnder diesseitiger Faktor in der wissenschaftlichen Erkenntnis. Aus dieser Einstellung folgt eine gewisse Zirkularität in unserem Bemühen, unsere Welt zu erkennen. Das begriffskritische Problem, das sich in Hinsicht auf die allgemeinen "Fragen der Sinnhaftigkeit unserer Sprachausdrücke und der Objektbezogenheit unserer Begriffsbildung" (p. 15, Bernays) stellt, ist deshalb grundsätzlich nicht lösbar. "Damit gelangen wir zu einer existenziellen Haltung, die man als diejenige einer erkenntnistheoretischen Demut bezeichnen kann" (p. 359).

Angewandt auf die Mathematik, bedeutet diese Einstellung ein ambivalentes Verhältnis zu der inhaltlichen Auffassung der Mathematik. Auf der einen Seite ist diese Auffassung nicht eigentlich zu halten, wegen des zweifachen Versagens der Evidenz: "des Zusammenbruchs des intersubjektiven Einverständnisses der Mathematiker und der mengentheoretischen Antinomien" (p. 7). Auf der anderen Seite kann die inhaltliche Auffassung der Mathematik nicht völlig verlassen werden, weil alle Versuche auf die inhaltliche Auffassung zu verzichten schon aus technischen Gründen dazu verurteilt sind, die inhaltliche Auffassung doch wieder zu benutzen.

Die allgemeine Tendenz dieses Buches ist, "das Bewusstsein um die intimen Zusammenhänge, welche zwischen den Problemen des mathematischen Denkens und denen unserer allgemeinen geistigen Orientierung bestehen, nachdem es für lange Zeit abhanden gekommen war, wieder in höherem Grade wach zu machen. Ausgehend von den aktuellen Grundlagenproblemen der Mathematik, führt die Darlegung zu einer Perspektive, von der aus diese Problematik als nicht spezifisch für die Mathematik, vielmehr als grundsätzlich verbunden mit den allgemeinsten erkenntnistheoretischen Problemen verbunden erscheint" (p. 15, Bernays).

Das methodische Fazit dieses Buches ist, dass eine gewisse Zirkularität in unseren erkenntnistheoretischen Gedankengängen nicht zu vermeiden ist.

Interessant ist des Autors Lösung der mengentheoretischen Antinomien. "In der Mengenlehre können die verschiedenen Bedeutungsgewebe, in die der mengentheoretische Existenzbegriff eingebettet ist, in Konflikt miteinander geraten. Es liegen dort zwei derartige Bedeutungsgewebe vor, das deduktiv-logische Gewebe, das in der deduktiven Struktur einer formalen Mengenlehre teilweise formal erfasst wird, und dasjenige der zulässigen Mengenbildungen, welches den Mengenbegriff in einen solchen Zusammenhang einordnet, in dem Existenzfeststellungen nicht auf deduktiven Wege, sondern sozusagen apodiktisch, durch direkte Konzipierung ("Bildung") der Menge, gemacht werden. Diese beiden Gewebe sind grundsatzlich heterogen. Diese beiden Zusammenhänge können nun, wie sich in den Antinomien zeigt, in Konflikt miteinander geraten. In Wirklichkeit sind aber in beiden Fällen ganz verschiedene gedankliche Prozesse involviert. Deren Konflikt ist nur unzulässig und skandalös, wenn wir ihn auf den Hintergrund einer an sich seienden, natürlich in sich widerspruchsfreien Realität, projizieren. Betrachten wir sie aber lediglich als das was sie sind: nämlich als gewisse Denkweisen und Konzeptionen des Menschen, so können wir in diesen Denkweisen nur eine gewisse Divergenz feststellen, die sich geltend macht, wenn man sie überspannt. Das ist für uns Anlass zu einer gewissen Vorsicht, in der Wertung unserer Begrifflichkeit. Dies ist aber auch alles" (pp. 321–323).

Das Buch ist in einem glänzenden philosophischen Stil geschrieben, der gelegentlich sogar dem literarischen Stil bedenklich nahekommt, was vielleicht dadurch erklärbar ist, dass der Author dieses Buch für einen grösseren Leserkreis bestimmt hat. Im grossen und ganzen, eine bemerkenswerte Leistung, die den Grundlagenmathematiker, sowie den allgemeinphilosophischen Erkenntnistheoretiker sehr interessieren dürfte. Aber auch der Laie dürfte aus diesem Buch eine allgemeine Orientierung über die zeitgenössische Philosophie der Wissenschaft und Mathematik schöpfen.

B. Germansky (Berlin)

2269:

Sternfeld, Robert. Philosophical principles and technical problems in mathematical logic. Methodos 8 (1956), 269–288.

The author argues "that mathematical logic is philosophical in nature and as such has the same character as all other branches of philosophy. It undergoes continual reformulation of its problems upon the occasion of the application of any new set of philosophical principles." Unfortunately, as the discussant of the paper, R. McNaughton, points out, the author's defense of his thesis is based on a confusion.

E. W. Beth (Amsterdam)

2270:
★**Martin, Gottfried. Klassische Ontologie der Zahl.** Kantstudien, Ergänzungshefte, 70. Kölner Universitäts-Verlag, Cologne, 1956. 159 pp.

This is the first volume of a work estimated at three volumes. It contains the historical development of the problem of the ontology of number. It covers the time from Pythagoras until Husserl and contains chapters, besides these two philosophers, also about Plato, Aristotle, Euclid, Descartes, Newton, Leibniz, Kant, Gauss, Weierstrass, Dedekind, Helmholtz, to quote only the more important names. The second volume will contain the ontological problematics of the investigation of our present days, beginning with Cantor and Frege, and the third volume the systematical representation of the problem.

The mathematical problems discussed historically are, e.g.: The reducing of certain classes of numbers to other classes of numbers; the question whether the mathematician invents or creates the mathematical figures and signs or only investigates them; the question whether there is necessary a philosophical understanding of number; the ideal number, the mathematical number and the sensual number; the question, whether there are ideas for all numbers; the significance of abstraction in mathematics; the problem of the concept of unity; the problem of the constructive character of Euclid's Elements; and so on.

The representation is clear and transparent. We await with interest the continuation of this work.

B. Germansky (Jerusalem)

2271:
Mal'cev, A. I. On classes of models which possess the operation of generation. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 738–741. (Russian)

The author introduces into the theory of models the following two notions. (1) A class K of models is said to be a class with natural generators if the intersection of the class of K-submodels of a given K-model M is either empty or is a K-submodel of M. (2) A class K of models is said to be pseudoaxiomatizable under the following conditions: a) if each finite part of some system of axioms, written with the predicate and individual symbols of the class K, is satisfied by a suitable K model, then the whole system is satisfied by a suitable K-model; and b) for each cardinal number $\mathfrak{m}$ there exists a cardinal number $\mathfrak{n}(\mathfrak{m})$ such that in each K-model which contains a set S of power $\mathfrak{m}$, there exists a K-submodel of power $\mathfrak{n}(\mathfrak{m})$ containing the elements of S. The present paper is an investigation of the relation among these notions and associated notions of the theory of models and abstract algebra. The results furnish conditions that a pseudoaxiomatizable class K with natural generators be axiomatizable.

E. J. Cogan (Bronxville, N.Y.)

2272:
Zubieta R., Gonzalo. Arithmetical classes defined without equality. Bol. Soc. Mat. Mexicana 2 (1957), 45–53. (Spanish)

This paper presents conditions under which a class K of models is axiomatizable in a system F with equality if and only if it is axiomatizable in an associated system F^* without equality. To provide these conditions, the author defines the relation of similarity between models and the equivalence class of a model $\mathfrak{A}$ under this relation. For a given class K, the class $C(K)$ is defined as the union of equivalence classes of models $\mathfrak{A}$ in the class K. The main result states that a class K is axiomatizable in F^* if and only if it is axiomatizable in F and $C(K)\subseteq K$. Similar results are proved for $\mathfrak{u}$-axiomatizability and finite $\mathfrak{u}$-axiomatizability. The final section provides other sets of conditions for K to be $\mathfrak{u}$-axiomatizable and finitely $\mathfrak{u}$-axiomatizable in F^*.

{Errata: p. 48, 4°(iii), a left-hand parenthesis is missing; p. 49, 2 lines before Theorem 4.3 and in (v) of the proof of that theorem, A should be $\mathfrak{A}$; p. 49, last line, $\mathfrak{A}$ should be $\bar{\mathfrak{A}}$.}

E. J. Cogan (Bronxville, N.Y.)

2273:
Nolin, Louis. Sur un système de "déduction naturelle". C. R. Acad. Sci. Paris **246** (1958), 1128–1131.

The author presents a system of natural deduction for the propositional calculus which is distinguished by the following features. (i) The segments involved have only single formulae in their succedents. (ii) Both axioms and derivations take the form of sequences of segments. It is stated that the theory of deduction of the lower predicate calculus can be developed in a similar way.

A. Robinson (Jerusalem)

2274:
McCluskey, E. J., Jr. Minimization of Boolean functions. Bell Tel. System Tech. Publ. Monograph **2720** (1956), 1–28.

2275:
Schröter, Karl. Die Vollständigkeit der die Implikation enthaltenden zweiwertigen Aussagenkalküle und Prädikatenkalküle der ersten Stufe. Z. Math. Logik Grundlagen Math. 3 (1957), 81–107.

This is a study of complete systems of axioms for the propositional calculus and for the pure lower predicate calculus on the basis of a particular choice of connectives. While there are many papers on the subject with regard to the propositional calculus, little interest has been shown in the problem in connection with the calculus of predicates. In the present paper the author produces a complete system of axioms and rules of deduction for the predicate calculus (as well as for the propositional calculus) which involve implication as their only connective, in conjunction with the propositional constant F (False). The proof of semantical completeness depends on reduction to standard systems.

A. Robinson (Jerusalem)

2276:
Klaua, Dieter. Ein Aufbau der Mengenlehre mit transfiniten Typen, formalisiert im Prädikatenkalkül der ersten Stufe. Z. Math. Logik Grundlagen Math. 3 (1957), 303–316.

This paper presents a formalization of a novel set theory with types in which both sets and classes appear. There are two primitive relations. The relationship of either set or class membership is expressed by $\in$, and the ordering relationship on the universe of discourse induced by the types associated with members of the universe is expressed by $\lesssim$. A member a of the universe is an "element" if $\exists x\ a \in x$, an "individual" if

$$a \text{ Element } \wedge \sim\exists x\ x\in a \wedge \forall x\ a \lesssim x,$$

a "class" (Klasse) if $\sim a$ Individual, and a set (Menge) if a Element $\wedge$ a Klasse.

The first axiom asserts that the relation $\lesssim$ is a complete ordering of the universe as well as a well-ordering in the sense that every non-empty class has a member of least type; that there is no element of greatest type; that every member of a class is of type less than the type of

the class; and that for everything which is not of least type and which is of type less than a given class, there is a member of the class of type at least as great (i.e. no type can exist between that of a class and its members). The second axiom asserts, for any property of elements, the existence of the corresponding class of elements. The third axiom asserts the existence, for any element, of the set of things of smaller type. From the first and third axioms follows that the subclass of any set is a set. Axioms for infinity, choice and Ersetzung are the remaining axioms of the theory.

A number of results are proven in the theory, including results for well-ordered classes and equivalent formulations of the axiom of choice.

P. C. Gilmore (Yorktown Heights, N.Y.)

2277:

Thiele, Helmut. Vollständigkeit im Stufenkalkül. Z. Math. Logik Grundlagen Math. 3 (1957), 211–224.

These are variations on a well-known theme of L. Henkin [J. Symbolic Logic **15** (1950), 81–91; MR **12**, 70]. The results obtained differ from Henkin's in some details, e.g., the author considers also systems in which the axiom of extensionality does not hold.

A. Robinson (Jerusalem)

2278:

Törnebohm, Håkan. On two logical systems proposed in the philosophy of quantum-mechanics. Theoria, Lund 23 (1957), 84–101.

The main purpose of this paper is to present an efficient method of characterizing and comparing three-valued propositional logics. In analogy with two-valued logic, three-valued logic is characterized by its connectives: negation (unary); conjunction, disjunction, implication, and equivalence (binary). In the open system the connectives are unspecified; closure is obtained by specifying them. The closure may yield more than one connective of any of the foregoing types. The unary connectives are defined as sets of three constants (truth values). The binary connectives are defined as sets of three functions of one propositional variable whose values are truth values. The specification of these constants and functions provides closure and determines the logic. Conditions upon the specification of the constants and functions constitute closure conditions (e.g., the requirement that the three-valued system be an extension of the standard two-valued system and the requirement that a connective have a certain logical property such as commutativity).

This method is used to examine the formal properties of two three-valued logics, one due to Mme. P. Destouches-Février and the other by Prof. H. Reichenbach. The authors of these systems intended them for application to quantum mechanics. The philosophical arguments for these applications are mentioned but not discussed.

W. Salmon (Providence, R.I.)

2279:

Badillo Barallat, M. C. Automatization of syllogisms in a polyvalent logic. Calc. Automat. y Cibernet. **5** (1956), no. 14, 1–10. (Spanish. English summary)

2280:

Guillaume, Marcel. Rapports entre calculs propositionnels modaux et topologie impliqués par certaines extensions de la méthode des tableaux sémantiques. Système de Feys-von Wright. C. R. Acad. Sci. Paris **246** (1958), 1140–1142.

This paper is concerned with Feys' System t, which Sobociński has shown to be equivalent to von Wright's System M. Generalising the construction of a semantic tableau as described by the reviewer, the author establishes the completeness of this system with respect to a certain topological interpretation.

E. W. Beth (Amsterdam)

2281:

Shoenfield, J. R. The class of recursive functions. Proc. Amer. Math. Soc. **9** (1958), 690–692.

Theorem: The predicate "α is recursive" can be written in the form $(Ex)(y)(Ez)R(\alpha, x, y, z)$ with R recursive but cannot be written in the form $(x)(Ey)(z)R(\alpha, x, y, z)$ with R recursive.

If N (the set of natural numbers) is supplied with the discrete topology and N^N is supplied with the product topology, this latter space contains the nonrecursive functions as a dense subset; the proof is then effected with aid of a category argument.

R. M. Baer (Berkeley, Calif.)

2282:

Davis, Martin. The definition of universal Turing machine. Proc. Amer. Math. Soc. **8** (1957), 1125–1126.

In a previous note [Automata studies, Princeton, 1956, pp. 167–175; MR **18**, 103] the author gave a definition of universal Turing machine. That definition is satisfied by machines requiring possibly more than one computation to produce a single answer. In the present note a new definition, circumventing the situation, is proposed. It is proved that a Turing machine satisfying the new definition also satisfies the old. The converse does not hold.

R. M. Baer (Berkeley, Calif.)

2283:

Maehara, Shôji. Another proof of Takeuti's theorems on Skolem's paradox. J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1958), 541–556.

Verf. gibt einen neuen Zugang zu Takeutis beweistheoretischen Versionen des Löwenheim-Skolemschen Satzes. Er beginnt mit der Skizze einer Verallgemeinerung des Gentzenschen WF-Beweises der Zahlentheorie, die zum "principal theorem" führt: Eine rein-logische Sequenz, die zahlentheoretisch beweisbar ist, ist rein-logisch beweisbar (oder: ein rein-logisches Axiomensystem, das rein-logisch konsistent ist, ist zahlentheoretisch konsistent). Hieraus folgert er direkt Takeutis "first theorem", dass ein rein-logisches Axiomensystem, das nach Hinzufügung von Formeln, die ausdrücken, dass es unendlich viele Individuen gibt, konsistent ist, auch konsistent bleibt, wenn noch Formeln hinzugefügt werden, die die Abzählbarkeit ausdrücken. Auch für Takeutis "second theorem" [J. Math. Soc. Japan **9** (1957), 71–76, 192–194; MR **19**, 4, 829] wird ein neuer Beweis aus dem "first theorem" gegeben. Schliesslich wird gezeigt, wie man vom "second theorem" wieder zum "principal theorem" des Verf. zurückgelangt.

P. Lorenzen (Kiel)

2284:

Lenz, Hanfried. Zur Axiomatik der Zahlen. Acta Math. Acad. Sci. Hungar. **9** (1958), 33–44.

Der Verfasser schwächt die Peanoschen Axiome der natürlichen Zahlen folgendermassen ab. Definition: Ein Zahlensystem ist eine Menge S mit einer Abbildung π in sich von den folgenden Eigenschaften. (I) π ist umkehrbar, d.h. aus $x^\pi = y^\pi$ folgt $x=y$. (Das Bild des Elements x bzw. der Menge X ist hier mit x^π bzw. X^π bezeichnet.) (II) Es gibt ein Element $O \in S$, so dass aus $O \in U \subseteq S$ und $U^\pi = U \cap S^\pi$ stets $U=S$ folgt (Induktionsaxiom).

Dem Induktionsaxiom allein genügen Systeme ver-

schiedener Arten. Der Verfasser beweist, dass, bis auf eine Isomorphie, allein die drei ersten dieser Systeme, nämlich die natürlichen Zahlen, die ganzen Zahlen und die zyklischen Gruppen, den Axiomen (I) und (II) genügen.

Ausgehend davon lässt sich die Addition und Multiplikation in diesen drei Bereichen in natürlicher Weise gemeinsam einführen. Diese Einführung dürfte dem Standpunkt der neueren Algebra "am besten gerecht werden." *B. Germansky* (Berlin)

2285:

Rose, Alan. Many-valued logical machines. Proc. Cambridge Philos. Soc. **54** (1958), 307–321.

The author constructs, for $m=\sum_{i=1}^{n-1} a_i 2^{i-1}+2^{n-1}+1$ ($a_i=0, 1$), an m-valued propositional calculus based on 2-valued propositional calculus by defining a proposition Y as an n-tuple $(X_1, \cdots, X_n)$ of 2-valued propositions X_i and by assigning to Y the value $y=\min[m, 1+x_1+\cdots+x_n 2^{n-1}]$ when X_i is assigned the values x_i $(=0, 1)$. Mechanisms simulating this construction may be rigged to record a set [sets] of (x_i) which yield a given value [values] y. There are numerous diagrams which facilitate the discussion, as well as two examples illustrating applications. *R. M. Baer* (Berkeley, Calif.)

SET THEORY

2286:

Kondô, Motokiti. Sur l'uniformisation des ensembles nommables. C. R. Acad. Sci. Paris **246** (1958), 2712–2715.

In descriptive set theory, the problems concerning separation and uniformization are very important. In particular, according to Kondo [Proc. Imp. Acad. Tokyo **15** (1939), 193–199; see p. 198] every plane CA-set E is uniformizable by a CA-set, i.e., there exists a "uniform" CA-set M such that $M\subseteq E$ and $\mathrm{proj}_x M=\mathrm{proj}_x E$. Several theorems of various authors are deducible from this result. Pursuing his previous researches concerning relative descriptive set theory [same C. R. **242** (1956), 1841–1843, 1945–1948, 2084–2087, 2209–2212, 2275–2278; MR **17**, 933; **18**, 2], the author in the present note examines also uniformization problematics and outlines the proofs of some fundamental results, which were announced earlier without proofs by P. Novikov [Trudy Mat. Inst. Steklov **38** (1951), 279–316; MR **14**, 234] and J. W. Addison [Bull. Amer. Math. Soc. **63** (1957), 397]. Theorem 1: Every (P_n, K, K)-namable set is uniformizable by the difference of two (P^n, K, K)-namable sets. According to the author, (Theorems 2, 3) if $n\geqq 2$ it is legitimate to replace in the preceding wording the symbol P^n by P_n and both P^n, P_n by $P_n{}^n$, respectively. Theorem 1 implies the following Theorem 4: Two disjoint (P_n, K, K)-namable sets are separable by two disjoint $(P_n{}^n, K, K)$-namable sets (for the terminology and notations see reviews we quoted above).

Definition: The projection of (S, K_0, K)-namable sets for $K_0\subseteq K$ and its complements are called (P^1, K_0, K)-namable and (P_1, K_0, K)-namable, respectively; projections and complements of these sets are called (P^2, K_0, K)-namable and (P_2, K, K)-namable respectively, etc. $(P_n{}^n; K_0, K)$ means to be both (P^n, K_0, K)-and (P_n, K_0, K)-namable. *Đ. Kurepa* (Zagreb)

2287:

Iséki, Kiyoshi. A generalisation of a theorem of W. Sierpiński. Proc. Japan Acad. **34** (1958), 28.

Sierpiński's argument in [Gaṇita **5** (1954), 113–116; MR **19**, 4] applies as well to any infinite cardinal. *L. Gillman* (Princeton, N.J.)

2288:

✱Sierpiński, Wacław. Cardinal and ordinal numbers. Polska Akademia Nauk, Monografie Matematyczne. Tom 34. Państwowe Wydawnictwo Naukowe, Warsaw, 1958. 487 pp.

This book contains all the basic theorems about cardinals and ordinals, plus many more specialized results, some of which have not appeared before in a book. It may be regarded as a greatly expanded version of the author's "Leçons sur les nombres transfinis" [Gauthiers-Villars, Paris, 1928]; however, the vast amount of detailed information in the present work comes to several times that included in the earlier one. According to the foreword, the manuscript was completed in 1952, so that only a few of the results obtained since that date could be incorporated.

Professor Sierpiński writes in his usual kindly, patient style which makes reading a pleasure (in spite of the fact that the English, an anonymous translation from the Polish manuscript, is not polished). Mathematical prerequisites are nil, so that the work can be read and appreciated by the beginning student as well as by the advanced mathematician. A welcome feature is the inclusion of hundreds of exercises, the more difficult of which are accompanied by detailed solutions. Another is the citation (with references) of many results whose proofs are too lengthy to be included. There is a bibliography of about 250 titles. The index is too sketchy to be of service, but the table of contents is very useful, as it includes all the section headings. Misprints in the mathematical text are insignificant.

Chapter I: "Sets and elementary set operations". Algebra of sets. Sets are constructed "gradually", first from objects that are not sets, then from such objects and sets already formed, and so on. The author is content with classical terminology and notation, e.g., using "sum" for what is now more commonly designated "union". II: "Equivalent sets". Finiteness is defined in terms of the natural numbers. Ramsey's decomposition theorem is quoted (without proof). "Effective" equivalence is introduced: two sets are effectively equivalent if "we can establish" a one-one correspondence between them. {There seems to be some doubt as to what this means.} The Cantor-Bernstein theorem is presented, with the "ancestor-descendant" proof. {It would have been informative to include Zermelo's elegant proof.} III: "Denumerable and nondenumerable sets". Effective denumerability, e.g., of the set of algebraic numbers. Various definitions of "finite". IV: "Sets of the power of the continuum". Sets effectively of the power of the continuum. Space-filling curves. Effective decomposition of the set of natural numbers into a continuum of almost disjoint sets. V: "Comparing the power of sets". Continuum hypothesis. Russell paradox.

VI: "Axiom of choice" (44 pp.). Opinions about the meaning and significance of the axiom. Hilbert's ε-function. Mostowski's results on sets of fixed finite cardinal — e.g., the axiom of choice for 4-element sets is equivalent to the axiom of choice for 2-element sets. Applications of the axiom of choice: equivalence of various definitions of "finite"; denumerable union of denumerable sets; König's theorem. VII: "Cardinal numbers and operations on them". VIII: "Inequalities for

cardinal numbers". Typical exercise: Prove without the aid of the axiom of choice that if $\mathfrak{m}$ is a cardinal number such that $2^{\mathfrak{m}}\geqq\aleph_0$, then $2^{\mathfrak{m}}\geqq 2^{\aleph_0}$. Tarski's relation $\mathfrak{m}\leqq^{*}\mathfrak{n}$ (meaning that $\mathfrak{m}=0$ or every set of power $\mathfrak{n}$ is a union of $\mathfrak{m}$ nonempty disjoint sets). IX: "Difference of cardinal numbers". (Results obtainable without the axiom of choice.) Quotient of cardinal numbers. Quoted without proof: If a cardinal number is divisible by 2 and by 3, then it is divisible by 6. X: "Infinite series and infinite products of cardinal numbers". In accordance with the author's unhurried approach, sums and products are defined first for sequences, and then, later, for arbitrary families. XI: "Ordered sets" (37 pp.). Total order, partial order. Detailed examination of the order types ω (natural numbers), η (rationals), and λ (reals). XII: "Order types and operations on them" (36 pp.). Arithmetic of order types, including over 50 exercises on sums and products. Infinite sums and products. Segments, remainders; divisors; comparability. XIII: "Well-ordered sets".

XIV: "Ordinal numbers" (97 pp.). Burali-Forti paradox. Arithmetic of ordinals. Prime components (i.e., ordinals indecomposable with respect to addition); prime factors. Normal form. Epsilon-numbers. Sherman's theorem on right distributivity. Roots of ordinals. The equations $\alpha+\beta=\beta+\alpha$, $\alpha\beta=\beta\alpha$, and $\alpha^{\beta}=\beta^{\alpha}$. Hessenberg natural sum and product. XV: "Number classes and alephs" (41 pp.). Typical exercise: Prove without the aid of the axiom of choice that $2^{\aleph_1}\geqq\aleph_1{}^{\aleph_0}$ Continuum hypothesis. Detailed discussion of the set of denumerable ordinals. Initial ordinals. Arithmetic of alephs. Regular and singular numbers. Inaccessible alephs. XVI: "Zermelo's theorem and other theorems equivalent to the axiom of choice". Hartog's cardinal $\aleph(\mathfrak{m})$. Trichotomy (comparability). Well-ordering theorem. Tarski's arithmetic theorems. Zorn's theorem (or Hausdorff's maximal principle). Teichmüller's theorem (also referred to in the literature as Tukey's lemma). Theorems of Kurepa and Vaught. Inference of the axiom of choice from the generalized continuum hypothesis. XVII: "Applications of Zermelo's theorem". Hamel basis. Decomposition into almost disjoint sets, and other decomposition theorems. Cofinal and coinitial subsets of ordered sets; η_α-sets and universal-order sets, including Mendelson's new proof of universal order (in an appendix).

The reviewer has a number of minor criticisms. There are so many misprints in the bibliography and in the references thereto that one wonders whether these portions were proofread. Examples: the reader is asked (on p. 262) to consult p. 165 of a paper listed as occupying pp. 72–75; "Miller" appears as "Müller" in the bibliography (and both ways in the text); the entry Sierpiński [26] should not duplicate [27] but should refer instead to Fund. Math. **28** (1937), 115–119.

Some of the references are open to question. If credit for the generalization of J. König's theorem is considered necessary, it ought to include Jourdain as well as Zermelo, and probably also Hausdorff [Math. Ann. **65** (1908), 435–505; p. 494, footnote]. The result attributed on p. 385 to Denjoy, 1953, was proved by Dushnik [Bull. Amer. Math. Soc. **37** (1931), 860–862], who, in turn, cites an earlier proof by Alexandroff and Urysohn. Since Leśniewski's name does not appear in the statement of Theorem 1 on p. 414, the casual reader may be misled by the gratuitous reference to Iseki. The statement that Hausdorff formulated the maximal principle as early as 1914 is relegated to a footnote.

Finally, the author could be more agressive in asserting his authority, less willing to accept the role of mere reporter. The brief discussion on inaccessibility (p. 405–406) simply lists several of the early definitions; the reader is not guided by being told which of these is in current use (or informed of the author's personal preference), and the established term "strongly inaccessible" is not even mentioned. It seems extravagant to state König's theorem four times and to omit mention of the efficient symbol $cf(\alpha)$ and the consequent compact expression

$$\aleph_\alpha{}^{\aleph_{cf(\alpha)}}>\aleph_\alpha.$$

The maximal principle is passed over as just another result in a long list of theorems equivalent to the axiom of choice: its practical utility is not pointed out, and it is never applied in a proof; those mathematicians trained in its use as a standard technique will read (p. 188) with surprise that the theorem on extending a partial order to a total order is a difficult one.

None of these matters, however, detracts seriously from the immense value of the work as a compendium of interesting mathematical information, presented with care and clarity. *L. Gillman* (Princeton, N.J.)

2289:
Kapuano, Isaac. Le problème restreint du continu et une conjecture de M. Denjoy. C. R. Acad. Sci. Paris **246** (1958), 33–36.

"Propositions relatives au problème restreint du continu et qui se démontrent par des choix non dénombrables. L'axiome du choix peut toutefois être évité si l'hypothèse suivante est vraie: il existe une loi faisant correspondre à tout ordinal μ de deuxième classe et de deuxième espèce une suite simple bien déterminée d'ordinaux μ_i qui tendent en croissant vers μ." (Résumé de l'auteur) *L. Gillman* (Lafayette, Ind.)

ORDER

See also 2274, 2393, 2402.

2290:
Šik, František. Automorphismen geordneter Mengen. Časopis Pěst. Mat. **83** (1958), 1–22. (Czech and Russian summaries)

The author examines the mutual relations between any ordered set M and any group Γ of its automorphisms. Fourteen theorems are proved, involving elementary considerations of group theory. All the automorphisms of M form an l-group G, where for $f, g\in G$ one obviously defines $f\vee g$, $f\wedge g$ by $(f\wedge g)x=fx\wedge gx$, $(f\wedge g)x=fx\wedge gx$, respectively. Given M, Γ, a point x of M is stable provided $fx=x$ for every $f\in\Gamma$; the set of all non-stable points is denoted $M(\Gamma)$. Γ is ordered in a natural way by $f\leqq g\Leftrightarrow f(x)\leqq g(x)$ $(x\in M)$. For $x\in M$, $f\in\Gamma$, the x-cycle of f is the union of all the [)-intervals of M whose endpoints are of the form $f^{(n)}(x)$ $(n=0, \pm1, \cdots)$. A cycle is proper if it has more than one point. Γ is monocyclic provided every $f\in\Gamma$ has at most one proper cycle. If f has a proper cycle C then every $g\in G$ such that $g=f$ in C and $g=1$ in $M\setminus C$ is called a phase of f. Γ is said to have the α-property, symbolically $\Gamma\in(\alpha)$, provided for every $f\in\Gamma$, $f\neq1$, there exist a phase g of f and an integer $n\neq0$ such that $g^{n}\in\Gamma$. Γ is divergent if for every $x, y\in M(\Gamma)$ with $x<y$ there

exists an $f \in \Gamma$ satisfying $fx \geqq y$. If $f_0, f_1 \in G$, then f_0, f_1 have the same orientation provided $f_i x > x \Rightarrow g_{i-1} x \geqq x$ for $i=0, 1$ and for each $x \in M$. Th. 1: Let Γ be transitive in $M(\Gamma)$; then Γ is simply ordered if and only if the conjugate elements in Γ have the same orientation. Th. 3: The following statements are mutually equivalent: a) Γ is monocyclic; b) Γ is divergent, simply ordered and archimedean; c) Γ is simply ordered and $\Gamma \in (\alpha)$. Th. 5, 6, respectively: The order-type $\tau(\omega^*+\omega)$ [resp. $\omega^*+\omega$] is characterized as belonging to any M with a jump if there exists a Γ which is transitive simply ordered [and archimedean]. Th. 8: If Γ is transitive and has one of the properties a)-c) stated in Th. 3, if M has at least 2 points and has a jump [neither a jump nor a gap], then the order-type of Γ is $\omega^*+\omega$ [λ] and is the same as that of M.

Đ. Kurepa (Zagreb)

2291:

Matsushima, Yataro. The geometry of lattices by *B*-covers. Proc. Japan Acad. **33** (1957), 328–332.

Let L be a lattice. The B-cover, B^*-cover and the $B^\dagger$-cover of two elements a and b of L are defined by $B(a, b)=\{x|x=(a\cap x)\cup(b\cap x)=(a\cup x)\cap(b\cup x),\ x \in L\}$, $B^*(a, b)=\{x|abx, x \in L\}$, $B^\dagger(a, b)=\{x|bax, x \in L\}$, where abx means $b \in B(a, x)$. The author continues his earlier work on B-covers [same Proc. **32** (1956), 549–553; Sci. Rep. Gunma Univ. **2** (1952); MR **18**, 713], and characterises several properties in lattices by B-, B^*- ,and $B^\dagger$-covers.

Ph. Dwinger (Lafayette, Ind.)

2292:

Matsushima, Yataro. On *B*-covers and the notion of independence in lattices. Proc. Japan Acad. **33** (1957), 462–467.

In this paper the author continues his earlier work on B-covers [see review above and references therein]. Let L be a lattice and let a and b be any two elements of L. The J-cover of a and b is defined by $J(a, b)=\{x|(a\cap x)\cup(b\cap x)=x, x \in L\}$. The CJ-cover is defined by $CJ(a, b)=\{x|(a\cup x)\cap(b\cup x)=x, x \in L\}$. A relative modular pair $(a, b)M^*$ is a pair of elements a and b of L such that $a\cap b \leqq x \leqq b$ implies $(x\cup a)\cap b=x\cup(a\cap b)$. The author shows that there exists a close connection between J-covers or CJ-covers and the relation of relative modularity. He also investigates the relations between the B-covers and independent sets in lattices.

Ph. Dwinger (Lafayette, Ind.)

2293:

Szász, G. On complemented lattices. Acta Sci. Math. Szeged **19** (1958), 77–81.

The author gives some proofs of known results on relatively complemented lattices.

R. P. Dilworth (Pasadena, Calif.)

2294:

Avann, S. P. A numerical condition for modularity of a lattice. Pacific J. Math. **8** (1958), 17–22.

For each a in a finite modular lattice L, the number $M(a)$ of x such that $x\cup a$ covers x and a is equal to the dual number $N(a)$ of y such that $y\cap a$ is covered by y and a. The author defines a finite lattice as near-modular when $M(a)=N(a)$ for all a. He studies the class of such near-modular lattices, showing that it is closed under direct product.

G. Birkhoff (Cambridge, Mass.)

2295:

Grätzer, G.; and Schmidt, E. T. On the lattice of all join-endomorphisms of a lattice. Proc. Amer. Math. Soc. **9** (1958), 722–726.

The authors refine a remark of the reviewer to the effect that the lattice of join-endomorphisms of an arbitrary lattice need not be semi-modular. They show that the lattice of join-endomorphisms of a finite lattice is semi-modular if and only if it is distributive, and, moreover, it is distributive if and only if the lattice is distributive.

R. P. Dilworth (Pasadena, Calif.)

2296:

Grätzer, G.; and Schmidt, E. T. On ideal theory for lattices. Acta Sci. Math. Szeged **19** (1958), 82–92.

Some new proofs are given for known theorems on the relations between prime ideals and distributive lattices [see J. Hashimoto, Math. Japon. **2** (1952), 149–186; MR **15**, 192].

R. P. Dilworth (Pasadena, Calif.)

THEORY OF NUMBERS

See also 2334, 2437, 2556, 2860, 3048.

2297:

Kanold, Hans-Joachim. Ein Satz über zahlentheoretische Funktionen. Math. Nachr. **18** (1958), 36–38.

For any positive integers r and n, let $\sigma_r(n)=\sum_d d^r$; $\varphi_r(n)=\prod_p (p^r-1)/p^r$, where d ranges over all divisors and p over all prime divisors of n. Denote by $M(x)$ the number of positive integers $m \leqq x$ which are prime to either $\sigma_r(m)$ or $\varphi_r(m)$ or both. In this paper it is shown that $\lim_{x\to\infty} M(x)/x=0$.

R. J. Levit (San Francisco, Calif.)

2298:

Nicol, C. A.; and Vandiver, H. S. Supplement to a paper entitled "A Von Sterneck arithmetical function and restricted partitions with respect to a modulus". Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 917–918.

The arithmetical function referred to in the title is $\Phi(k, n)=\phi(n)\mu(n/D)/\phi(n/D)$, where D is the greatest common divisor of n and k, and ϕ and μ are the functions of Euler and Möbius. The authors give reasons why $\Phi(k, n)$ should be called the "Dedekind-Von Sterneck number". The sum of the kth powers of the primitive nth roots of unity (Ramanujan's sum) is equal to $\Phi(k, n)$ and the authors feel this should be called the "Dedekind-Hölder theorem".

T. M. Apostol (Pasadena, Calif.)

2299:

Schönheim, Ioan. Formules pour résoudre la congruence $x^2 \equiv a$ (mod P) dans des cas encore inconnus et leur application pour déterminer directement des racines primitives de certains nombres premiers. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. Fiz. **7** (1956), no. 1–4, 51–58. (Romanian. Russian and French summaries)

Consider the prime modulus $P=2^\alpha h+1$, $\alpha>2$, h odd. Let $e>1$ be the odd factor of the exponent to which 2 belongs for this modulus P. Let p_i be the prime factors of e, and define $e_i=e/p_i$. Then it is shown that the congruences $x^2 \equiv (-1)^{(p_i-1)/2} p_i$ (mod P) have the solutions $x \equiv \pm(1+2\sum_{\mu_i} 2^{2^{\alpha-1} e_i \mu_i})$ respectively, where the μ_i run over the quadratic residues of the p_i.

I. A. Barnett (Cincinnati, Ohio)

2300:

Clarke, P. J. General tests for divisibility. Math. Gaz. **42** (1958), 122–123.

Let n_k and n_k' be the solutions of $10^k n_k \equiv 1$ and $10^k \equiv n_k'$ (mod n) having smallest absolute value. Both n_k and n_k' are tabulated for prime $n \leqq 47$ and $k \leqq 10$. These are used

in divisibility tests which depend on the congruences $n_k(10^kA+B)\equiv A+n_kB$ and $10^kA+B\equiv n_k'A+B \pmod{n}$.

L. Moser (Edmonton, Alta.)

2301:

Leech, John. On $A^4+B^4+C^4+D^4=E^4$. Proc. Cambridge Philos. Soc. **54** (1958), 554–555.

Using EDSAC II, the author finds all solutions to the equation in the title, for $|E|\leqq 4303$, a limitation imposed by the word length of the machine. The computation, although clearly finitary, is reduced a million-fold by the use of 16 and 5 as exclusion moduli; for instance, it then follows that three of the four quantities A, B, C, D are divisible by 2, and another three of them are divisible by 5, while E is divisible by neither (for a primitive solution). The smallest of eight primitive solutions found is the quintuple (30, 120, 272, 315, 353), discovered by Norrie [Univ. St. Andrews 500th Anniv. Mem. Vol., Edinburgh, 1911, pp. 87–89].

H. Cohn (Tucson, Ariz.)

2302:

Schönheim, Ioan. Détermination d'une solution de l'équation $\sum_{j=1}^N x_j^2=N\prod_{j=1}^N x_j$ pour un entier $N>2$ quelconque, x_j étant des nombres premiers deux à deux. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. Fiz. 7 (1956), no. 1–4, 59–63. (Romanian. Russian and French summaries)

The author shows that the recurrence

(i) $$a_n=a_1a_2\cdots a_{n-1}+k,$$

where the a_i and k are relatively prime in pairs, defines numbers a_i which are also relatively prime in pairs, Formula (i) is equivalent to (ii) $a_n=a_{n-1}^2-ka_{n-1}+k$, $n>2$, and (iii) $a_2=a_1+k$. Furthermore, $x_j=a_{j+1}$ is the solution of the diophantine equation $\sum_{j=1}^N x_j^2=N\prod_{j=1}^N x_j$, for an arbitrary $N>2$, where the x_j are relatively prime in pairs. He utilizes the sequence defined by (i) to generalize the proof of Pólya and Szegö for the existence of an infinite number of primes.

I. A. Barnett (Cincinnati, Ohio)

2303:

Mordell, L. J. On the Kusmin-Landau inequality for exponential sums. Acta Arith. 4 (1958), 3–9.

Soient a_ν réels, $e^{ix}=e(x)$, $S=\sum_1^n e(2a_\nu)$, $\Delta a_{\nu-1}=a_\nu-a_{\nu-1}$, $0<\theta\leqq\Delta a_1\leqq\Delta a_2\leqq\cdots\leqq\Delta a_{n-1}\leqq\varphi<\pi$. Les estimations pour $|S|$ ont été données par van der Corput, Kusmin et Landau dans les cas $\varphi=\pi-\theta$ [J. F. Koksma, Diophantische Approximationen, Springer, Berlin, (1936); p. 115]. J. Karamata et le rapporteur ont donné des estimations analogues, en supposant seulement $\varphi<\pi$ [Acad. Serbe Sci. Publ. Inst. Math. 3 (1950), 207–218; MR 12, 482]. La plupart de ces résultats on obtient par des considérations géométriques élémentaires. L'auteur montre que toutes ces estimations résultent déjà de la transformation du type d'Abel $\sum_1^n \lambda_\nu(\mu_\nu-\mu_{\nu+1})=\sum_1^n \mu_\nu(\lambda_\nu-\lambda_{\nu-1})$, avec $\lambda_0=\mu_{n+1}=0$. En plus, on obtient ainsi des résultats plus précis, par exemple, de $\varphi\leqq\pi/2$, $S'=\frac{1}{2}e(2a_1)+\sum_2^{n-1} e(2a_\nu)+\frac{1}{2}e(2a_n)$ il vient $|S'|\leqq\operatorname{cotg}\theta$. L'auteur déduit, dans le même ordre d'idées, le résultat suivant dû à I. Popken [Koksma, loc. cit., p. 115]: de $0<\theta\leqq\Delta a_r\leqq\pi/4$ $(r=1, 2, \cdots, n-1)$, $\Delta^2a_r=\Delta(\Delta a_r)\geqq 0$ $(r=1, 2, \cdots, n-2)$, $\Delta^3a_r=\Delta(\Delta^2a_r)$ $(r=1, 2, \cdots, n-3)$, résulte $|S|\leqq 1/\sin\theta$.

M. Tomić (Belgrade)

2304:

Turán, P. On the so-called density-hypothesis in the theory of the zeta-function of Riemann. Acta Arith. 4 (1958), 31–56.

Let $N(T)$ be the number of zeros $\rho=\beta+i\gamma$ of $\zeta(s)=\zeta(\sigma+it)$ with $0<\gamma\leqq T$, and $N(\alpha, T)$ the number of these with $\beta\geqq\alpha$. The density hypothesis (DH) may be stated with various degrees of refinement. The roughest form is

$$N(\alpha, T)<AT^{(2+\varepsilon)(1-\alpha)}\log^B T\ (\tfrac{1}{2}\leqq\alpha\leqq 1,\ T\geqq C),$$

where A, B are absolute constants, ε is an arbitrarily small positive number, and $C=C(\varepsilon)$. (In this review ε and C will always be used thus, and may have different values at different occurrences.) With $B=5$ this was shown by the reviewer to follow from the Lindelöf hypothesis (LH) that $\zeta(\frac{1}{2}+it)=O(t^\varepsilon)$ as $t\to\infty$ [Quart. J. Math. Oxford Ser. 8 (1937), 255–266]. The object of this paper is to deduce this hypothesis, with $A=1$, $B=0$, from the weaker conjecture (B): If (i) $\frac{1}{2}<\frac{1}{2}+10\delta\leqq\kappa\leqq\alpha<1$, and (ii) the rectangle $\alpha\leqq\sigma\leqq 1$, $|t-\tau|\leqq[\log\frac{1}{2}\tau]$ contains no zero of $\zeta(s)$, then, for $\tau>\tau_0(\kappa, \delta)$, the square $\alpha-\delta\leqq\sigma\leqq\alpha$, $|t-\tau|\leqq\frac{1}{2}\delta$, contains at most $\delta g(\delta)\log\frac{1}{2}\tau$ zeros, where $g(\delta)\downarrow 0$ as $\delta\downarrow 0$. This is, in fact, weaker, since LH implies, even without (ii), that the number of zeros in the square is $o(\log\tau)$ uniformly in α as $\tau\to\infty$. An intermediate conjecture (A) is B without the restriction (ii).

The starting point is an identity

$$\sum_\rho f(\rho-s)=\sum_n \Lambda(n)F(n)n^{-s}+R(s),$$

proved by contour integration. Here, f is defined by

$$f\left(\frac{z}{\lambda}\right)=\left(e^z\frac{\sinh\varepsilon z}{\varepsilon z}\right)^k\ (k \text{ a positive integer}),$$

with $\lambda=\lambda(\varepsilon)>0$ and $\varepsilon\lambda$ large; the corresponding $F(n)$ is 0 outside a finite interval; and $R(s)$ is relatively unimportant. Consideration of the mean value of $|\sum_n|^2$ (and so of $|\sum_\rho|^2$) over $T\leqq t\leqq 2T$, for $\frac{1}{2}<\sigma<1$, leads to a dissection of the strip $T\leqq t\leqq 2T$ into two sets of horizontal strips $I=I(T, \alpha)$, $J=J(T, \alpha)$, relative to an arbitrary α in $\frac{1}{2}+\varepsilon\leqq\alpha\leqq 1-\varepsilon$. The set I is relatively small (in total width and in the number of strips comprising it) and contains only $o(T^{2(1-\alpha)})$ zeros altogether. The set J is such that $|\sum_\rho|<CT^{\alpha-\sigma+\varepsilon}$ when k belongs to a certain interval $K(T)$ and s belongs to J and to a certain discrete subset $S(T)$ of the strip $\frac{1}{2}<\sigma<1$. If the inequality for $\sum_\rho$ could be transferred to the individual terms, it would follow that any ρ in J must have $\beta<\alpha+\varepsilon$. The author seeks to approximate to this situation, for a suitable k, by first restricting $\sum_\rho$ to a set of ρ near s, and by then applying his general method based on sums of powers of complex numbers [P. Turán, Eine neue Methode in der Analysis und deren Anwendungen, Akadémiai Kiadó, Budapest, 1953; MR 15, 688]. This, however, is not completely effective; but, by an elaborate analysis of the possible distribution of zeros in J, the difficulty is reduced to one that can be resolved by the assumption of conjecture B with a suitable $\delta=\delta(\varepsilon)$. The result is an inequality for $N(\alpha+\varepsilon, 2T)-N(\alpha+\varepsilon, T)$, from which DH is deduced. For α near 1, however, the deduction involves another deep theorem of the author (Satz XXXVIII of the book quoted above). {It is not clear how the refinement $A=1$, $B=0$ is obtained by the method of p. 52, which seems to ignore the factor $\log^6 T$ in (1.1.10); it can, however, be obtained with the help of deep theorems on the non-existence of zeros in regions of the type $1-\eta(t)\leqq\sigma\leqq 1$.}

In support of his contention that B is essentially weaker than LH or A, the author proves in an appendix the corresponding statement with $g(\delta)$ replaced by 0.71, and remarks that a similar (or more drastic) modification of A seems almost as difficult as LH itself.

A. E. Ingham (Cambridge, England)

2305:

Bateman, Paul T.; and Grosswald, Emil. On a theorem of Erdös and Szekeres. Illinois J. Math. 2 (1958), 88–98.

Let h be a given integer >1, and let $N_h(x)$ denote the number of positive integers $n \leqq x$ which have the property that $p^h | n$ for every prime factor p of n. It was proved by Erdös and Szekeres [Acta Sci. Math. Szeged **7** (1934), 95–102] that $N_h(x)=\gamma_0 x^{1/h}+O(x^{1/(h+1)})$, where γ_0 is a constant. The present authors derive more precise information. If $h>2$, they show that $N_h(x)=\gamma_0 x^{1/h}+\gamma_1 x^{1/(h+1)}+O(x^{1/(h+2)})$. If $h=2$, when the generating Dirichlet series is $\zeta(2s)\zeta(3s)/\zeta(6s)$, they replace the O-term by $O(x^{1/6}\exp(-a\omega(x)))$, where $\omega(x)=(\log x)^{4/7}(\log\log x)^{-3/7}$. In the case $h>2$ they also give an estimate with $r+1$ major terms instead of two, where r is the largest integer $<(2h)^{\frac{1}{2}}$. These results are obtained by writing the generating Dirichlet series as a product of a series belonging to a lattice point problem (and which is a product of ζ's) and a series with a relatively small abscissa of absolute convergence.
N. G. de Bruijn (Amsterdam)

2306:

Palamà, G. Sul problema analogo a quello di Waring. Matematiche, Catania **11** (1956), 117–120 (1957).

By considering the system of multigrade congruences $a_1, a_2, \cdots, a_n \stackrel{m}{\equiv} b_1, b_2, \cdots, b_n \pmod{p}$, the author shows that the multiples of infinitely many primes occurring in the arithmetic progression $(m+1)n+1$, $n=1, 2, \cdots$, are representable by means of $2(m+1)$ of the mth powers $\pm 1^m, \pm 2^m, \pm 3^m, \cdots$.
W. H. Simons (Vancouver, B.C.)

2307:

Holzer, Ludwig. Über eine modifizierte Schnirelmann-Summe. Math. Nachr. **18** (1958), 298–308.

Let A, B be sets of positive integers both containing 0. Let α, β be the Schnirelmann densities of A and B. Put $\mu=\alpha+\beta$, $\sigma=\alpha/(1-\beta)$. The author considers the set $G=G_n$ of integers $a+b \leqq n$ such that if $a \neq 0$ then $b<n/2$. There exist sets A, B such that $G(x)<\sigma x \leqq (\alpha+\beta)x$. The author first finds conditions under which $G(x) \geqq \sigma x$. His results show in particular that $G(x)>\sigma x$ if $2\alpha+\beta \leqq 1$. In the second part of his paper the author finds conditions under which $G(x) \geqq \mu x$.
H. B. Mann (Columbus, Ohio)

2308:

Iseki, Shô. Some transformation equations in the theory of partitions. Proc. Japan Acad. **34** (1958), 131–135.

The author announces a transformation formula for the generating function of $p(n; a, M)$, the number of partitions of n into positive summands of the form $Ml \pm a$ ($l=0, 1, 2, \cdots$), where $0<a<M$, $(a, M)=1$, $M \geqq 2$. A detailed proof, based on a functional equation derived in an earlier paper [Duke Math. J. **24** (1957), 653–662; MR **19**, 943] is promised in a future paper. The special case $a=1$, $M=4$, which is treated in some detail, leads to a simplification of Hua's results [Trans. Amer. Math. Soc. **51** (1942), 194–201; MR **3**, 270] concerning the generating function for the number of partitions of n into odd (or unequal) parts.
T. M. Apostol (Pasadena, Calif.)

2309:

Kohlbecker, Eugene E. Weak asymptotic properties of partitions. Trans. Amer. Math. Soc. **88** (1958), 346–365.

The author proves two Abelian-Tauberian theorems, both involving slowly oscillating functions (i.e., continuous positive functions $L(x)$ satisfying $L(cx)/L(x) \to 1$ as $x \to \infty$, for every $c>0$).

(i) If $n(u)$ is positive and non-decreasing, if $g(s)=s\int_0^\infty e^{-su}/(1-e^{-su})n(u)du$ and if $\alpha>0$, then the conditions $n(u) \sim u^\alpha L(u)$ $(u \to \infty)$ and $g(s) \sim \Gamma(\alpha+1)\zeta(\alpha+1)s^{-\alpha}L(s^{-1})$ $(s \downarrow 0)$ are equivalent. (If we omit the factors $(1-e^{-su})^{-1}$ and $\zeta(\alpha+1)$ this reduces to a theorem of Karamata [J. Reine Angew. Math. **164** (1931), 27–39].)

(ii) If $P(u)$ is positive and non-decreasing, $f(s)=s\int_0^\infty P(u)e^{-su}du$, and if $\alpha>0$, $A>0$, then the conditions

$$\log f(s) \sim A\alpha^{-1}s^{-\alpha}L(s^{-1}) \quad (s \downarrow 0)$$
$$\log P(u) \sim (1+\alpha^{-1})u^{\alpha/(1+\alpha)}L_1(u) \quad (u \to \infty)$$

are equivalent. Here L_1 is a slowly oscillating function which has the following property. Let, for large values of u, the number s_u satisfy $u=As_u^{-\alpha-1}L(s_u^{-1})$; then $s_u \sim u^{-1/(\alpha+1)}L_1(u)$ $(u \to \infty)$. It is shown that s_u and L_1 exist if L is given.

Theorems (i) and (ii) can be combined by taking $g(s)=\log f(s)$, $A=\alpha\Gamma(\alpha+1)\zeta(\alpha+1)$. As a result, asymptotic formulas can be obtained for the logarithm of certain partition functions. For example, if $0<\lambda_1<\lambda_2<\cdots$, and if $n(u)$ is the number of λ_i which do not exceed u, then $P(u)$ denotes the number of solutions of $m_1\lambda_1+m_2\lambda_2+\cdots \leqq u$ in integers $m_i \geqq 0$. The case with $L(u) \equiv 1$ is due to Knopp [Schr. Königsberg. gelehrt. Ges. Naturw. Kl. **2** Heft 3 (1925), 45–74]. {In Theorem 2, $(e^{-su}-1)$ should be $(1-e^{-su})$.}
N. G. de Bruijn (Amsterdam)

2310:

Delange, Hubert. On some arithmetical functions. Illinois J. Math. **2** (1958), 81–87.

The author proves the following statement of P. Erdös: Let $\omega(n)$ be the number of prime divisors of the positive integer n, and let λ be any irrational number. Then the numbers $\lambda\omega(n)$ are uniformly distributed modulo 1. This means that for $0 \leqq t \leqq 1$, the number of n's less than or equal to x and such that $\lambda\omega(n)-I[\lambda\omega(n)] \leqq t$ is $tx+o(x)$ as x tends to $+\infty$ ($I[n]$ denotes the greatest integer $\leqq n$). It is pointed out that this result can be deduced, if not simply, from a later result of Erdös concerning the number of integers $n \leqq x$ for which $\omega(n)=k$ [Ann. of Math. (2) **49** (1948), 53–66; MR **9**, 333] and also that a very short proof can be based on a formula of Atle Selberg [J. Indian Math. Soc. (N.S.) **18** (1954), 83–87; MR **16**, 676], which, however, requires the properties of the Riemann zeta function in the critical strip.

The result of Erdös and the similar result for the total number of prime divisors of n are proved to be immediate consequences of a well-known theorem of H. Weyl [Math. Ann. **77** (1916), 313–352, Satz 1], a theorem of the author [Ann. Sci. Ecole Norm. Sup. (3) **73** (1956), 15–74, § 2.3; MR **18**, 720] and a Tauberian theorem [D. W. Widder, The Laplace transform, Princeton, 1941; Th. 17; MR **3**, 232] as formulated in the present paper. In fact, the uniform distribution modulo 1 of $\lambda f(n)$ for any irrational λ holds for all functions of a certain family $(\mathfrak{F})$ [Delange, Colloque sur la théorie des nombres, Bruxelles, 1955, Thone, Liège, 1956, pp. 147–161; MR **19**, 17] and also for n running through a certain infinite set of positive integers, distinct from the set of all positive integers.
S. Ikehara (Tokyo)

2311:

Delange, Hubert. Sur certaines fonctions arithmétiques. C. R. Acad. Sci. Paris **246** (1958), 514–517.

Let $\omega_E(n)$ denote the number of distinct prime divisors of the positive number n which belong to a set E of prime numbers, and let $\Omega_E(n)$ be the total number of prime divisors of n belonging to E. The author has studied

distribution properties of these functions [see review above and references therein] under the assumption (H): There is a constant α such that $\sum_{p\in E} p^{-s}+\alpha\log(s-1)$ is regular throughout the closed halfplane $Rs\geqq 1$. This is now replaced by a more general and natural assumption (H_1): There is a number $\alpha\geqq 0$ such that the number of numbers of E not exceeding x will be equal to $\alpha(x/\log x)+o(x/\log x)$ as x tends to $+\infty$, and if $\alpha=0$, we have $\sum_{p\in E} p^{-1}=+\infty$.

Previous results on arithmetical functions in the papers above may be all deduced under the new assumption, and a very general theorem is stated which includes the former results and which is susceptible of other applications. *S. Ikehara* (Tokyo)

2312:

Delange, Hubert. Sur la distribution de certains entiers. C. R. Acad. Sci. Paris **246** (1958), 2205–2207.

Continuation of paper reviewed above, indicating how other results may also be generalized under the new condition (H_1), including results of B. Hornfeck [Monatsh. Math. **60** (1956), 96–109; MR **18**, 18] and of E. Wirsing [Arch. Math. **7** (1956), 263–272; MR **18**, 642]. *S. Ikehara* (Tokyo)

2313a:

Petersen, G. M. 'Almost convergence' and uniformly distributed sequences. Quart. J. Math. Oxford Ser. (2) **7** (1956), 188–191.

2313b:

Keogh, F. R.; Lawton, B.; and Petersen, G. M. Well distributed sequences. Canad. J. Math. **10** (1958), 572–576.

Let $I=[a, b]$ be a subinterval of $[0, 1]$, $I(x)$ the characteristic function of I. A sequence $0\leqq s_n\leqq 1$, $n=1, 2, \cdots$ is called well-distributed if $I(s_n)$ almost converges to $b-a$ for each I, i.e., if $(I(s_{n+1})+\cdots+I(s_{n+p}))/p\to b-a$ for $p\to\infty$, uniformly in n. Well-distributed sequences form a subclass of uniformly distributed ones. In the first paper it is shown that s_n is well-distributed if and only if $\exp(2\pi iks_n)$ almost converges to zero for $n\to\infty$ and each $k=1, 2, \cdots$. Examples of sequences which are or are not well-distributed are given in the second paper. (1) t_k is well-distributed if s_k is and if $s_k-t_k\to 0$. (2) If θ is irrational, there is a sequence of positive integers n_k, $n_{k+1}/n_k>\lambda>1$, for which the sequence of fractional parts $\{n_k\theta\}$ is well-distributed. (3) If r is rational, the sequence $\{r^k\theta\}$ is not well-distributed for any θ. {Reviewer remarks that also the sequence $\{f(k)\}$ is well-distributed, where $f(x)$ is a polynomial with an irrational leading coefficient.} *G. G. Lorentz* (Syracuse, N.Y.)

2314:

Leveque, William J. On the frequency of small fractional parts in certain real sequences. Trans. Amer. Math. Soc. **87** (1958), 237–261.

Let $X_1, X_2, \cdots$ be a sequence of independent random variables, each uniformly distributed on $[0, \frac{1}{2}]$. Then if $f(k)$ is an arbitrary function of the positive integer k, whose values lie in $[0, \frac{1}{2}]$, the equation $\Pr\{X_k<f(k)\}=2f(k)$ holds, and from the Borel-Cantelli lemmas [cf. W. Feller, Probability theory and its applications, Wiley, New York, 1950; MR **12**, 424], it follows that the probability that the inequality (1) $X_k<f(k)$ is satisfied for infinitely many k is zero or one, according as (2) $\sum_{k=1}^{\infty} f(k)<$ or $=\infty$. If one puts (3) $U_k=$distance $\langle kx\rangle$ between kx and its nearest integer ($k\geqq 1$, x a random real variable), the sequence $U_1, U_2, \cdots$ is a sequence of dependent random variables, uniformly distributed on $[0, \frac{1}{2}]$. A well-known theorem of A. Khintchine [Math. Ann. **92** (1924), 115–125] shows that nevertheless the above statement concerning (1) remains true if one replaces X_k by U_k. The author investigates whether the U_k resemble the X_k also in their finer structure. He considers the case where (2) diverges and finds a result which is not quite what one might expect from the case of independent variables. If however in (3) the sequence $\langle kx\rangle$ is replaced by $\langle r_1r_2\cdots r_kx\rangle$, where $r_1<r_2<\cdots$ is a fixed increasing sequence of positive integers, the sequence $U_1, U_2, \cdots$ is much less strongly dependent and the results are in accordance with the situation in case of independency.

The results would take too much space to be quoted here in full. Moreover the author pointed out in a letter that he discovered an error in the last sentence of p. 246, which forces him to replace in Theorem 1 (the case (3)) the quantity T_n by

$$U_n=\mathrm{No}\{m\leqq n|\langle mx\rangle<g(m),\ (m, l_m)=1\},$$

where $l_m=l_m(x)$ is the integer nearest to mx. There will appear a further paper correcting and extending the above indicated results. *J. F. Koksma* (Amsterdam)

2315:

Mori, Mitsuya. Über Kummersche Erweiterungen. Proc. Japan Acad. **33** (1957), 372–375.

The author describes the well-known theory on Kummer extensions in terms of group characters and tensor products of groups, extending the idea of Iwasawa [J. Math. Soc. Japan **5** (1953), 253–262; MR **15**, 937] and Kawada [Duke Math. J. **22** (1955), 165–177; MR **16**, 907]. *Y. Kawada* (Tokyo)

2316:

Popovici, Constantin P. Sur l'unicité de la décomposition en facteurs premiers dans les anneaux des entiers de Dirichlet. Acad. R. P. Romîne. Stud. Cerc. Mat. **8** (1957), 73–101. (Romanian. Russian and French summaries)

The author continues his study [see Rev. Univ. "C. J. Parhon" Politehni Bucureşti. Ser. Şti. Nat. **3** (1954), 47–54; MR **17**, 947] of the ring $\mathfrak{J}[i, \theta]$ of integers of Dirichlet, of the relative quadratic field $R(i, \sqrt{\delta})$, where δ is a Gaussian integer without square factor. The main result of the paper is that there exist relative quadratic fields with unique decomposition into primes, and others where this decomposition is not unique. Actually, an integer that cannot be represented as a product of two integers, neither of which is a unit, is called indecomposable, the term "prime" being reserved for indecomposable integers p with the property that $p|ab$ and $p\nmid a \Rightarrow p|b$. There are three types of indecomposable Dirichlet integers: (a) proper divisors of Gaussian primes (these are "primes"); (b) Gaussian primes that have no proper divisors in $\mathfrak{J}[i, \theta]$; and (c) Dirichlet indecomposable integers, not Gaussian primes, but proper divisors of Gaussian integers, which are products of type (b) Dirichlet indecomposable integers. Clearly, if type (c) exists, the decomposition cannot be unique. Conversely, if type (c) is absent, the decomposition is unique, and the indecomposable integers of type (b) are "primes". This follows from the author's Theorem 5, which reduces the uniqueness of decomposition of Dirichlet integers to the uniqueness of decomposition of Gaussian integers into indecomposable Dirichlet integers. The proofs are relatively elementary, using, in general, Hasse's methods. The main result is Theorem 13, which states that the decomposition is unique, if and only if no Gaussian prime divisor of $x^2-\delta$

(where x runs through the Gaussian integers with $N(x) \leqq (N(\delta))^{\frac{1}{2}}$) is a Dirichlet indecomposable integer of type (b), except, in some cases, prime divisors of 4. For $N(\delta) \leqq 100$ a complete tabulation of all relative quadratic fields $R(i, \sqrt{\delta})$ is appended, where those of non-unique decomposition are marked.

E. Grosswald (Philadelphia, Pa.)

2317:

Popovič, Konstantin P. [Popovici, C. P.] On uniqueness of decomposition into prime factors in rings of quadratic integers. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) 1(49) (1957), 99–120. (Russian)

The author considers rings of all integers in the field of $\sqrt{d}$ or $(1+\sqrt{d})/2$ in the usual notation with d squarefree $\not\equiv 1$ or $\equiv 1$ mod 4 respectively. He develops a unique factorization theory presumably independent of ideal theory and applicable when no quadratic integer ω exists which is indecomposable, disregarding units, and fails to divide some rational prime. The familiar "Minkowskian-type" finiteness lemma nevertheless appears [taken from Gr. C. Moisil, Introducere in algebră. I. Inele şi ideale, vol. I, Editura Acad. R. P. Romîne, 1954; MR **16**, 990], to the effect that for given ω, all integers of the field are congruent (mod ω) to a finite set of $\omega_i = m + n\beta$, where $0 \leqq m < |N(\omega)/D|$, $0 \leqq n < |D|$, with discriminant $=D$. From this the author shows that the real field $R(\sqrt{d})$ posesses unique factorization if x^2-d (or $x^2-d/4$) for $0 \leqq x < \sqrt{d}$ has as prime divisors only those primes which are norms of integers in the quadratic field. There is a similar theorem for $d<0$ and several related results.

H. Cohn (Tucson, Ariz.)

2318:

Popovici, Constantin P. Critères de décomposition en facteurs premiers dans les anneaux imaginaires d'entiers quadratiques. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 9 (1957), 5–17. (Romanian. Russian and French summaries)

It is known that in a quadratic field $R((-p)^{\frac{1}{2}})$ with p a natural prime, the decomposition into primes can be essentially (i.e. except for units and the order of factors) unique, only if $p=2, 3, 7, 11$, or $p \equiv 19$ (mod 24). Previously, the author [#2317 above] has formulated a necessary and sufficient condition for uniqueness of decomposition, in terms of indecomposable quadratic integers of type (b) [see #2316 above]. In the present paper the author uses that result in order to obtain two criteria for essential uniqueness of decomposition in $R((-p)^{\frac{1}{2}})$, p a natural prime, $p \equiv 19$ (mod 24). These are: I) The decomposition into primes is essentially unique if and only if all integers x^2+p with integral x, $0<x<(p/3)^{\frac{1}{2}}$, are of the form $4^a P$ ($a=0, 1$; P prime); II) the decomposition into primes is essentially unique if and only if there are no quadratic residues $r \neq 2$, $r<p/4$.

E. Grosswald (Philadelphia, Pa.)

2319:

Barbilian, D. L'argument d'Euclide pour l'infinité des nombres premiers. Acad. R. P. Romîne. Stud. Cerc. Mat. 8 (1957), 7–72. (Romanian. Russian and French summaries)

The paper contains new proofs of many known theorems (some not in the strongest known version) and several results that appear to be new. The main interest lies in the unified method of proof, essentially an elaboration of Euclid's proof of the infinity of primes. The author gives credit to A. Châtelet for the idea of a specific way of using Euclid's argument [related to that of Wendt, J. Reine Angew. Math. **115** (1895), 85–88], while Châtelet himself suggests (without bibliographical references) that Hilbert was familiar with it. Using this argument, the proof that there are infinitely many primes in the arithmetic progression $p=1+nr$ ($r>1$, given integer; $n=0, 1, 2, \cdots$) runs as follows: Assume that the corresponding set of primes is finite, with m elements ($m<\infty$, but $m=0$ not excluded) and set $l=\lambda p_1 p_2 \cdots p_m{}^r$ ($l=\lambda r$, if $m=0$), where λ is a natural integer to be determined later. The cyclotomic polynomial $\Phi_r[x]$ has integral coefficients and ± 1 as last coefficient. Hence, setting (*) $\Phi_r[x]=x\Psi_r[x]\pm 1$, $\Psi[x]$ also has integral coefficients. As $\Phi_r[x]=\pm 1$ has only a finite number of roots λ may be selected so that $\pm\Phi_r[l]\neq 1$ and $|\Phi_r[l]|$ is a rational integer greater than one. Hence, there exists a natural prime $p>1$, with $p|\Phi_r[l]$. From this, the definition of l, and (*), follows $r \not\equiv 0$ (mod p). Consequently, (**) $x^r-1\equiv 0$ (mod p) has the same number of primitive roots as the equation $x^r-1=0$. From this and the choice of p it follows that l is a primitive root of (**), whence (***) $l\Psi[l]\pm 1\equiv 0$ (mod p). Consequently, $(l, p)=1$ and, by Fermat's theorem, $l^{p-1}-1\equiv 0$ (mod p). But, l being a primitive root of (**), the exponent is a multiple of r, or $p-1=nr$, $n>0$. This already shows that $m>0$. Furthermore, by assumption, $p=1+nr$ is one of the primes p_i ($1\leqq i\leqq m$). Hence, $l\equiv 0$ (mod p_i) and (***) implies $\pm 1\equiv 0$ (mod p_i); as $p_i>1$ the last congruence is impossible and the theorem is proven. Using this and other versions of Euclid's argument the author: (1) extends to non-commutative rings with unit element many theorems concerning decompositions into indecomposable elements (as in previously reviewed papers of C. Popovici [#2316-#2318 above], the term "prime" is used with a more restricted meaning) and gives sufficient conditions that these should form an infinite set; (2) gives an algebraic (but not elementary — using the theory of conductors, ramification of ideals, etc.) proof, that for any algebraic number field of degree n, there exist infinitely many prime ideals of degree $g<n$, and, for the case of cyclic fields of prime degree, infinitely many prime ideals of first degree; (3) gives the above sketched version of the proof that there are infinitely many primes in the arithmetic progression $nr+1$, and another proof that in quadratic fields there are infinitely many prime ideals of first degree; (4) gives an algebraic proof that there are infinitely many prime ideals of second degree in quadratic fields of negative (and some of positive) discriminant. Most results of (1), concerning non-commutative rings, seem to be new and interesting. However, their statement would require an excessively large number of definitions; therefore, only a few results will be quoted [for unfamiliar terms, see the author's Teoria aritmetica a idealelor, Editura Acad. R. P. Romîne, 1956; MR **19**, 11]. A) Let O be a semi-group with unit element, subcommutative on the right and such that: (i) each element is a product of only a finite number of factors; (ii) for each element to be prime on the left implies that it is indecomposable; (iii) the left cancellation law holds. Then every element of O can be factored into a finite number of indecomposable elements, uniquely determined up to the order and right association. B) An ordered ring (not necessarily commutative, or with unit element) contains no divisors of zero. C) If a ring has at most two units, then it can only be of characteristics 0, 2, 3, 4, or 6. D) Let Ω be a ring with unit element, of characteristic different from zero, and assume, furthermore (a) Ω is subcommutative on the right; (b) the indecomposable elements are not right divisors of zero; (c) the units are at most (± 1); (d) every element, not a

unit, has an indecomposable left divisor. Then the set of indecomposable elements of Ω is infinite.

E. Grosswald (Philadelphia, Pa.)

2320:

Mori, Mitsuya. Über die Klassenkörpertheorie für unendliche Erweiterungen von einem $\mathfrak{p}$-adischen Zahlkörper. Proc. Japan Acad. **33** (1957), 376–379.

Let k_0 be a $\mathfrak{p}$-adic number field and k an algebraic extension of k_0 of infinite degree. The theory of abelian extensions K/k was treated by Moriya [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **5** (1936), 9–66] under the assumption that the degree $[K:k]$ is relatively prime to the infinite part N_∞ of the formal degree $N=[k:k_0]$. The author proves an analogy of local class field theory over such a field k without any restriction on the degree of abelian extensions of k. The main tool is the fundamental group of k, which is defined as follows. Let $k=\lim k_n$, $k_0\subset k_n$, $[k_n:k_0]<\infty$, and let $\mathfrak{p}^*=\lim \mathfrak{p}_n$ where $\mathfrak{p}^*$ and $\mathfrak{p}_n$ are valuations of k and k_n respectively. Let $k_i^\wedge$ be the completion of the multiplicative group $k_i^\times$ in the sense of Artin [Algebraic numbers and algebraic functions, I, Inst. Math. Mech., New York Univ., 1951; MR **13**, 628], which is a compact group. Let $N_{\mu\nu}^\wedge(\mu<\nu)$ be the continuous norm-homomorphism of $k_\nu^\wedge$ into $k_\mu^\wedge$. The fundamental group $\mathfrak{F}(k)$ is defined to be the compact projective limit group of $\{k_\mu^\wedge\}$ with respect to $N_{\mu\nu}^\wedge$. This group $\mathfrak{F}(k)$ can be also defined directly from $k^\times$. Replacing the role of the multiplicative group $k_0^\times$ for k_0 in local class field theory by the group $\mathfrak{F}(k)$ for k, the main theorems in class field theory can be proved simply by limit processes. Finally, it is proved that the compact Galois group G of the maximal abelian extension A_k of k is topologically isomorphic to $\mathfrak{F}(k)$.

Y. Kawada (Tokyo)

2321:

Mahler, K. An interpolation series for continuous functions of a p-adic variable. J. Reine Angew. Math. **199** (1958), 23–34.

L'auteur donne une nouvelle et très intéressante démonstration du théorème d'approximation de Weierstrass pour une fonction continue à valeurs p-adiques, définie dans l'espace $\mathfrak{Z}_p$ des entiers p-adiques. Cette démonstration a l'intérêt de prouver que le processus d'approximation peut être pris linéaire par rapport à la fonction approché f: de façon précise, si on pose $a_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(n-k)$, la suite des nombres p-adiques a_n tend vers 0, et la série $\sum_{n=0}^{\infty}a_n\binom{x}{n}$ converge uniformément vers $f(x)$. L'auteur montre en outre que si f est continûment dérivable, alors toutes les séries $a_n'=\sum_{k=1}^{\infty}(-1)^{k-1}a_{k+n}/k$ sont convergentes, la suite (a_n') tend vers 0 et on a $f'(x)=\sum_{n=0}^{\infty}a_n'\binom{x}{n}$.

J. Dieudonné (Evanston, Ill.)

2322:

Endler, Otto. Differentiation in algebraischen Funktionenkörpern von n Variablen. Math. Z. **67** (1957), 413–427.

Let K be an n-dimensional function field over the ground field k. The author studies those valuations w of K whose residue field is of dimension $n-1$ over k. If K_0 is a subfield of K such that K is algebraic over K_0, then for each w the "differential exponent" $m_w(K/K_0)$ is defined as usual by means of the trace [see, e.g., Chevalley, Introduction to the theory of algebraic functions of one variable, New York, 1951; MR **13**, 64; Chap. IV, § 8]; it is $<+\infty$ if and only if K/K_0 is separable. The main object of this paper is to look for a connection, in the case where K_0 is generated by a separating basis of transcendency $x=(x_1,\cdots,x_n)$ of K, between $m_w(K/K_0)$ and the w-value of dx, where dx stands for the alternating differential form $dx_1\cdots dx_n$ of degree n, and its w-value $w(dx)$ is to be defined as follows. Write $dx=u\cdot dz$ with $u\in K$, where $z=(z_1,\cdots,z_n)$ is uniformizing for w in the sense that (i) z is a separating basis of transcendency of K, and (ii) the valuation ring of w is mapped into itself by the partial derivations D_{z_i} of K with respect to z. Then $w(dx)=w(u)+\mathrm{Min}_y\,(m_w(K/k(y)))$ where y ranges over all the bases of transcendency of K. (The minimum in the last formula is zero if k is perfect or, more generally, if the residue field of w is separable over k.) In the case of dimension $n=1$, the connection between $m_w(K/k(x))$ and $w(dx)$ is given by the classical formula of Riemann and Hurwitz which states that both are equal. However, in dimension $n>1$, things are not so simple, and the author can prove the formula $w(dx)=m_w(K/k(x))$ only in the case where the residue field of w is separable over k, and x satisfies some additional conditions with respect to w which can be stated as follows: (i) among the w-residues $\bar{x}_1,\cdots,\bar{x}_n$ there are $n-1$ which are algebraically independent with respect to k; (ii) there is an element x_0 in K such that the system $(x_0,x_1,\cdots,x_n)$ contains a uniformizing set for w. (The author then calls x "semiregular" with respect to w.) If (ii) is not fulfilled or if the residue field of w is not separable over k then the author can prove only a certain inequality which, as he says, is not a satisfying generalization of the Riemann-Hurwitz formula. He remarks that F. K. Schmidt in an unpublished paper has succeeded in giving the proper generalization of the Riemann-Hurwitz formula for higher dimensions.

P. Roquette (Hamburg)

2323:

Häuslein, Günter. Über die Modulfunktionen arithmetischer Körper höheren Grades. Math. Nachr. **16** (1957), 73–78.

Suppose that $K=k(x,y)$ is a field of algebraic functions of one variable with the defining (absolutely irreducible) equation $f(x,y)=0$ over the algebraic number field k of absolute degree g. Using the isomorphisms $k\to k^{(i)}$ into the field of all complex numbers C the author associates to K the function fields $CK^{(i)}/C$ by replacing the coefficients of $f(x,y)$ by their ith conjugates. The fields $K^{(i)}$ have the same genus p as K. Furthermore, denote by $\sigma_j^{(i)}$, $1\leq j\leq p$, half systems of retrosections on the Riemann surfaces of the $CK^{(i)}$, which can be augmented to canonical systems of retrosections. Now observe that the differentials of the first kind in each field $K^{(i)}$ form modules $M^{(i)}$ of rank pg over the ring of integers. Systems of generators $w^{(i)}$ for these modules are picked as follows: Let ϕ_j, $1\leq j\leq p$, be a k-basis of the module of differentials of the first kind in K/k. Denoting by $\Gamma=C_1Z+\cdots+C_gZ$, Z the ring of integers, the maximal order of K, Steinitz' theorem implies $M=\Gamma\phi_1+\cdots+\Gamma\phi_{p-1}+\mathfrak{b}\phi_p$, with an ideal $\mathfrak{b}=b_1Z+\cdots+b_gZ$ of Γ. Then the elements $W_1=C_1\phi_1,\cdots$, $W_{p-1}=C_1\phi_{p-1}$, $W_p=b_1\phi_p,\cdots$, $W_{p(g-1)+1}=C_g\phi_1,\cdots$, $W_{pg-1}=C_g\phi_{p-1}$, $W_{pg}=b_g\phi_p$ form a basis of M over Z. Passage to the fields $K^{(i)}$, replacing the coefficients of these expressions with respect to x, y and their multipliers by their conjugates, determines similar bases $w_\nu^{(i)}$ for the modules $M^{(i)}$. Finally set $A_j^{(i)}=((\sigma_\alpha^{(i)},w_\beta^{(i)}))$ and $Z(w,\sigma)=(A_j^{(i)})$ with $1\leq j\leq g$, $1\leq\alpha\leq p$, $(j-1)p+1\leq\beta\leq jp$, where $(\sigma,w)=\int_\sigma w$. With this notation the author shows, taking advantage of results of C. L. Siegel and Hel Braun on

modular forms, that $\sum |\det Z(w,\sigma)|^{-s}$, summation being taken over $\{\sigma^{(1)}, \cdots, \sigma^{(g)}\}$, complete sets of non-associated half-systems of retrosections, is formally a modular function which is analytic for $R(s)>p+1$.

O. F. G. Schilling (Chicago, Ill.)

2324:

Obrechkoff, Nikola. Sur l'approximation diophantienne des nombres réels. C. R. Acad. Sci. Paris **246** (1958), 31–32.

L'auteur démontre le théorème suivant: Soit a un nombre natural et soit n un nombre naturel plus grand que a. Alors pour chaque nombre réel ω, satisfaisant à la condition $0<\omega\leqq a$, il existe aux moins deux nombres entiers, x, y, non négatifs et tels qu'on ait

$$(1)\qquad |\omega x-y|\leqq\left\{\left[\frac{n-a}{a+1}\right]+2\right\}^{-1}\quad(0<x+y\leqq n).$$

Le signe d'égalité dans (1) est atteint.

La démonstration du théorème utilise les propriétés de la suite de Farey. *J. Popken* (Amsterdam)

2325:

Obrechkoff, Nikola. Sur l'approximation diophantienne des formes linéaires. C. R. Acad. Sci. Paris **246** (1958), 204–205.

An inequality is obtained concerning the approximation of a homogeneous linear form in variables x_μ, divided into p sets, with the condition that in each set the x_μ are of equal sign. The result is best possible.

C. G. Lekkerkerker (Amsterdam)

2326:

Sudan, Gabriel. Über das Gesetz der besten Näherung. Rev. Math. Pures Appl. 2 (1957), 429–433.

Let α be a real irrational number. The author calls a rational number p/q a best approximation to α if all rational numbers closer to α have larger denominators. A new proof is given of the following known theorem. [See J. F. Koksma, Diophantische Approximationen, Springer, Berlin, 1936, p. 37 ff., for literature.] A rational number p/q is a best approximation to α if and only if it is either a convergent p_n/q_n to α, or else a quasiconvergent $(p_nc+p_{n-1})/(q_nc+q_{n-1})$ with either $a_{n+1}<2c<2a_{n+1}$ or $a_{n+1}=2c$ and $[a_{n+1}, a_n, \cdots, a_2, a_1]>[a_{n+1}, a_{n+2}, \cdots]$.

W. J. LeVeque (Göttingen)

2327a:

Ehrhart, Eugène. Sur les polygones homothétiques. C. R. Acad. Sci. Paris **246** (1958), 205–207.

2327b:

Ehrhart, Eugène. Polygones homothétiques et inéquations diophantiennes linéaires. C. R. Acad. Sci. Paris **246** (1958), 354–357.

2327c:

Ehrhart, Eugéne. Sur les inéquations diophantiennes linéaires. C. R. Acad. Sci. Paris **246** (1958), 1147–1149.

Let S denote a plane region and let i, p denote the number of points with integral coordinates in the interior and on the boundary of S, respectively. Put

$$\Delta(S)=i+\tfrac{1}{2}p-V(S),$$

where $V(S)$ denotes the area of S. A polygon S is called "semi-entier" if all the coordinates of the vertices of S are $\equiv\frac{1}{2}$ or 1 (mod 1) and at least one is $\equiv\frac{1}{2}$ (mod 1). Then it is shown that, for a "semi-entier" polygon S, the value of $\Delta(S)$ is invariant under homothetic transformations of S (with respect to any fixed integral point), provided that the homothetic ratio is an odd integer.

In the second paper, polygons S with rational vertices are called "$1/n$-entier", if n is the l.c.m. of the denominators of the coordinates of these vertices. An analogue of the above invariance property is established and, in the third paper, application is made to diophantine inequalities of the form

$$\frac{X}{a}+\frac{Y}{b}<C,$$

where X, Y are positive integral variables, $a>0$, $b>0$ are coprime integers, and C is any real number.

J. H. H. Chalk (Hamilton, Ont.)

2328:

Linnik, Yu. V. Asymptotic-geometric and ergodic properties of sets of lattice points on a sphere. Mat. Sb. N.S. **43(85)** (1957), 257–276. (Russian)

A detailed proof of the results about the uniform distribution of integral points on the surfaces of spheres enunciated in Dokl. Akad. Nauk SSSR (N.S.) **96** (1954), 909–912 [MR **16**, 451]. The proof is on similar lines to that there adumbrated but is considerably simplified by using ergodic theory. *J. W. S. Cassels* (Cambridge, England)

COMMUTATIVE RINGS AND ALGEBRAS

See also 2341, 2344, 2345, 2356, 2357.

2329:

Morikawa, Hisasi. On the existence of unramified separable infinite solvable extensions of function fields over finite fields. Nagoya Math. J. **13** (1958), 95–100.

If K is an algebraic function field of one variable of genus >1 over the constant field k, which is finite and has at least 11 elements, there exists an unramified separable solvable extension of infinite degree of K which is regular over k. The proof uses previous results of the author [#2340 below] and an inequality, based on the Riemann hypothesis, whose application gives rise to the restriction on the cardinality of k.

M. Rosenlicht (Evanston, Ill.)

2330:

Baer, Reinhold. Algebraic closure of fields and rings of functions. Illinois J. Math. 2 (1958), 37–42.

Let R denote a ring of functions, from a set D to a field F, that contains the constants and separates the points of D. Two topological spaces are associated with R: (1) the set T of maximal ideals in R, with the closure of a set $S\subset T$ defined as the set of all ideals in T that contain the intersection of all the ideals in S (Stone topology); (2) the set D, with the collection of all sets $\{d\in D:f(d)=0\}$, for $f\in R$, taken as a base for the closed sets (zero-set topology). The mapping that takes each $d\in D$ into the maximal ideal $\{f\in R:f(d)=0\}$ is a homeomorphism of D into T. Let D^σ be the image of D in T.

The ring R is called a full ring of functions if D is compact and every nonvanishing $f\in R$ has an inverse in R. It is observed that $T=D^\sigma$ implies that R is full; and, as a partial converse, that if T is Hausdorff and R is full, then $T=D^\sigma$. The main result of the paper is the following theorem: $T=D^\sigma$ for every full ring of functions to F if and only if the field F is not algebraically closed.

C. W. Kohls (Urbana, Ill.)

2331:

Satô, Hazimu. A note on principal ideals. J. Sci. Hiroshima Univ. Ser. A. **21** (1957/58), 77–78.

The author gives proofs of the following two assertions, of which the first is a lemma stated in a paper by the reviewer [Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **29** (1955), 293–303], and the second a generalization of an assertion made in another paper by the reviewer [ibid. 59–77; MR **17**, 122; **20** 3857]:

(1) If $\mathfrak{p}$ is a prime divisor of a principal ideal aR, where R is Noetherian and a is an element of R which is not a zero divisor, then for any element c of $\mathfrak{p}$ which is not a zero divisor $\mathfrak{p}$ is a prime divisor of cR.

(2) Let a be an element of a Noetherian ring and assume that a is not a zero divisor. Let $\mathfrak{p}$ be a prime ideal of R containing a such that $R_{\mathfrak{p}}$ is not a discrete valuation ring. Then, for any element b of $aR:\mathfrak{p}$, b/a is integral over R.

M. Nagata (Cambridge, Mass.)

2332:

Rayner, F. J. Hensel's lemma. Quart. J. Math. Oxford Ser. (2) **8** (1957), 307–311.

It is shown how the known proof of Hensel's lemma for complete local rings can be adapted to complete pseudo-valuation rings. The result is as follows: Let R be a complete pseudo-valuation ring. Let $\mathfrak{m}$ and $\mathfrak{n}$ be ideals of R consisting of all elements of positive values and of value infinity, respectively. If for a polynomial $f \in R[x]$ there are g and h such that (i) $f \equiv gh$ ($\mathfrak{m}$), (ii) $(g, h, \mathfrak{m}) = (1)$ and (iii) the polynomial g is monic (i.e., the leading coefficient $=1$), then f modulo $\mathfrak{n}$ can be factored to the product of two polynomials G and H over $R/\mathfrak{n}$ which are congruent to g and h respectively modulo $\mathfrak{m}$, where G is a monic polynomial.

M. Nagata (Cambridge, Mass.)

2333:

Ogaĭ, S. V. On a method of solving a cubic equation. Kirgiz. Gos. Univ. Trudy Fiz.-Mat. Fak. **1953**, no. 2, 125–127. (Russian)

2334:

Rédei, Ladislaus. Zur Theorie der Polynomideale über kommutativen nullteilerfreien Hauptidealringen. Math. Nachr. **18** (1958), 313–332.

Given any principal ideal ring H with unit 1, let H_k denote the (commutative) polynomial ring $H[x_1, \cdots, x_k]$ $(k=1, 2, \cdots)$; thus, except when H is a field and $k=1$, H_k is not itself a principal ideal ring. In this paper effective procedures are introduced for determining when an ideal of H_k, given in terms of a finite number of generators $f_1, \cdots, f_r$, is principal. The author does not specify the precise meaning he attaches to the term "effective" in this context, but the euclidean algorithm is (reasonably enough) accepted, so that the problem reduces at once to the special case in which the f_i have no non-trivial common divisor, i.e., to determining effectively whether or not $(f_1, \cdots, f_r) = (1)$. In the affirmative case, effective procedures are given for solving $\sum f_i u_i = 1$ for $u_i \in R$. The case $k=1$ is dealt with in full detail, the general case being said to involve no additional difficulties.

In spite of related work of G. König [Einleitung in die allgemeine Theorie der algebraischen Größen, Teubner, Leipzig, 1903], the author claims his results to be new, even for H the ring of integers. Various special cases and more or less closely related problems (mainly of a number-theoretic nature) are discussed at length; in particular, a new and somewhat ponderous proof of the quadratic reciprocity law is presented.

M. P. Drazin (Baltimore, Md.)

2335:

Satô, Hazimu. Some remarks on Zariski rings. J. Sci. Hiroshima Univ. Ser. A. **20** (1956/57), 93–99.

Let A be a Zariski ring and $\hat{A}$ its completion. In § 1, an exact relationship between prime divisors of an ideal $\mathfrak{a}$ of A and those of $\mathfrak{a}\hat{A}$ is given. In § 2 the theorem of transition is proved; in § 3 the following are proved: (i) If every maximal chain of prime ideals in $\hat{A}$ has length equal to rank A, then so does every such chain in A; (ii) if $\hat{A}$ is a unique factorization ring, then so is A. {These results in §§ 2–3 in the case of local rings have been known before; the known proofs for local rings can be applied to Zariski rings without any change, and the author has done so substantially. The author gives a reference to a relevant article of the reviewer [Proc. international symposium on algebraic number theory, Tokyo and Nikko, 1955, pp. 191–226, Science Council of Japan, Tokyo, 1956; MR **18**, 637] on the theory of multiplicity (in Japanese), from which he could have learned of the known proofs. A reference to [6] should be given for (i) stated above. Footnote 4 gives the wrong source for the unique factorization theorem in unramified regular local rings: it is not Krull but Y. Mori who proved the theorem first (Mori announced the result in 1949 and at the same time he announced (ii) above in the case of local rings).}

M. Nagata (Cambridge, Mass.)

2336:

Beatty, S. Difference methods in the theory of local order bases and their equivalent normalized function bases. Trans. Roy. Soc. Canada. Sect. III. (3) **50** (1956), 1–11.

ALGEBRAIC GEOMETRY

See also 2322, 2736, 2745.

2337:

Horadam, A. F. A locus in [8] invariant under a group of order 51840×81. Quart. J. Math. Oxford Ser. (2) **8** (1957), 241–259.

The author considers a group of collineations of order 51840×81 in complex projective space of eight dimensions. This group contains 81 harmonic inversions, each with respect to an S_3, Σ and an S_4, Π. The author finds a locus L, of dimension 4 and order 45, invariant under the group; L contains each of the solids Σ, and meets any Π in a set of 45 points forming a Burkhardt configuration. {The author's Theorem 11, asserting that each Π contains 45 solids lying on L, seems to be based on a misunderstanding.}

J. A. Todd (Cambridge, England)

2338:

Pišl, Milan. Circular cubics and bicircular quartics. Pokroky Mat. Fys. Astr. **3** (1958), 32–41. (Czech)

Dans un système de coordonnées isotropes, l'auteur étudie les propriétés des tangentes, des points doubles, des foyers et des bitangentes d'une cubique circulaire et d'une quartique bicirculaire.

K. Svoboda (Brno)

2339:

Drăgilă, P. Sur la correspondance par parallélisme de deux surfaces. Proc. Amer. Math. Soc. **9** (1958), 189–200.

L'A. crede di dimostrare che sono falsi i classici risultati di Peterson, Guichard ed altri sulle superficie (dello spazio ordinario) che si corrispondono per parallelismo di normali. Il recensore è di avviso opposto. Il teorema

cui l'A. si riferisce è il seguente: "Si deux surfaces x, y, z et X, Y, Z se trouvent dans une relation telle que les normales aux points correspondants soient parallèles, alors par chaque point d'une surface passent deux directions qui sont parallèles (ou deux courbes dont les tangentes sont parallèles) aux directions correspondantes sur l'autre surface. Ces deux directions, sur chaque surface, sont conjuguées et par consequent, constituent sur les deux surfaces les réseaux auxquels nous avons donné le nom de base de la relation" [K. M. Peterson: Ann. Fac. Sci. Univ. Toulouse 2 ser. **7** (1905), 1–43; p. 26]. Negli esempi che, secondo l'A., dovrebbero provare la falsità dell'enunciato, si verifica semplicemente il fatto che le tangenti coniugate uscenti da ogni punto sono complesse coniugate, ciò che ovviamente non toglie nulla alla validità del teorema di Peterson.

V. Dalla Volta (Rome)

2340:

Morikawa, Hisasi. Generalized jacobian varieties and separable abelian extensions of function fields. Nagoya Math. J. **12** (1957), 231–254.

Lang has shown how the class field theory for unramified abelian coverings of an algebraic curve is related to the separable isogenies of the jacobian variety of the curve [Ann. of Math. (2) **64** (1956), 285–325; MR **18**, 672]. This paper first shows that any separable abelian covering of an algebraic curve may be obtained as a "pull back" of a separable isogeny of a commutative algebraic group onto a suitable generalized jacobian variety of the curve. This had been previously noted by Lang and proved by Serre [Groupes algébriques et théorie du corps de classes, Collège de France, 1957, (mimeographed)] by using the universal mapping property of generalized jacobians [see Rosenlicht, Ann. of Math. (2) **66** (1957), 80–88; MR **19**, 579]; here the result is obtained by first showing the existence of a separable homomorphism from a suitable generalized jacobian of the covering onto a generalized jacobian of the base curve. As a consequence, the author obtains (as does Serre, loc. cit.) a full class field theory à la Lang for arbitrary separable abelian extensions of a function field of one variable over a finite constant field.

M. Rosenlicht (Evanston, Ill.)

2341:

Rees, D. On a problem of Zariski. Illinois J. Math. **2** (1958), 145–149.

Let k be the field of complex numbers, (x_0, x_1, x_2) the homogeneous generic point of an elliptic cubic curve C, and let $A'=k[x_0, x_1, x_2, ty, tz, t^{-1}]$, where y and z are linear forms in x_0, x_1, x_2 which form a basis of the homogeneous ideal of a point P of C, and t is an indeterminate over $k(x_0, x_1, x_2)$. If A is the integral closure of A' in its quotient field $F=k(x_0, x_1, x_2, t)$, the author proves that A does not satisfy the Nagata conditions [Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1956), 57–70; MR **19**, 458] for F to be a field satisfying the conditions of Zariski's generalization of Hilbert's 14th problem [Bull. Sci. Math. (2) **78** (1954), 155–168; MR **16**, 398]. This answers negatively the Zariski conjecture [loc. cit.] for fields of transcendence degree $\geqq 3$.

P. Abellanas (Madrid)

2342:

Whitney, Hassler. Elementary structure of real algebraic varieties. Ann. of Math. (2) **66** (1957), 545–556.

Definitions: In real n-space R^n, or in complex n-space C^n, the differential $df(p)$ of a function f at a point p is a covector, defined in the usual way. The rank of a set S of functions at a point p is the maximum number of independent differentials $df_1(p), \cdots, df_s(p)$, the f's being in S. The rank of a point set Q at $p \in Q$ is the rank at p of the ideal formed by all the polynomials (with real or complex coefficients) which vanish in Q. The set of common zeros (in R^n or C^n) of a set of polynomials is a (real or complex) algebraic variety V. An algebraic partial manifold M is a point set associated with an integer ρ, such that for any $p \in M$ there exists a set of polynomials $f_1, \cdots, f_\rho$, of rank ρ at p, and a neighbourhood U of p, such that $M \cap U$ is the set of zeros of the f's in U; and $n-\rho$ is then the dimension of M.

For any V, let M_1 be the set of points $p \in V$ where the rank of V is its maximum. Then, both in the real and in the complex cases, it is proved that M_1 is an algebraic partial manifold, and that $V_1=V-M_1$ is void or is a proper algebraic subvariety of V. The "splitting process" leading from V to $M_1 \cup V_1$ can now be applied to V_1, if V_1 is not void, and comes to an end after a finite number of steps, giving

$$V=M_1 \cup M_2 \cup \cdots \cup M_s$$

(where $s \leq 2^{n-1}$); here the M's are algebraic partial manifolds two by two disjoint, and each point set $V_i= M_{i+1} \cup \cdots \cup M_s$ is an algebraic variety, and so is closed. The M's need not be closed (or connected), since each M_i may have limit points in later M's. A different splitting (with at most n terms) can be obtained by manifolds of decreasing dimension.

It is further shown that, in the real case, each M_i in the previous splitting has but a finite number of topological components. More generally it is proved that, if V' is a subvariety (possibly void) of a real variety V, then $V-V'$ has at most a finite number of topological components; this is obtained by studying the connection between V and the smallest complex variety containing V.

B. Segre (Rome)

2343:

Samuel, Pierre. Rational equivalence of arbitrary cycles. Amer. J. Math. **78** (1956), 383–400.

Let V^n be a non-singular variety in a projective space. An attempt to generalize the notion of linear equivalence of divisors to arbitrary cycles and put it in an elaborate form is given here. Let X^r be a V-cycle of dimension r. If there is a non-singular unirational variety R (a non-singular variety which is a rational image of a projective space), a cycle Z on $R \times V$ and two points a, b on R such that $X=Z(a)-Z(b)$, the author defines that "X is rationally equivalent to 0" and writes $X \sim 0$. The set $R^r(V)$ $[R_r(V)]$ of r-cycles [cycles of co-dimension r] forms an additive subgroup of the group $G^r(V)$ $[G_r(V)]$ of all r-cycles [cycles of co-dimension r]. By showing that any two points on a non-singular unirational variety can be joined by a rational curve, it is shown that R can always be replaced by a projective line. Next, it is proved that rational equivalence is compatible with fundamental operations on non-singular projective varieties, i.e., it is compatible with the operations of i) specialization, ii) intersection-product, iii) product and iv) algebraic projection. It is also shown that, when W is a non-singular subvariety of V, then $X \cdot W \sim 0$ on W if $X \sim 0$ on V and $X \cdot W$ is defined; and that when $X^r \sim 0$ on V, there is a V-cycle Y^{r+1} and a function f on V such that $X=(f) \cdot Y$.

The author then shows that multiplication between elements of $G_r(V)/R_r(V)$ and $G_s(V)/R_s(V)$ with values in $G_{r+s}(V)/R_{r+s}(V)$ is defined (which is induced by the operation of intersection-product) and that the direct sum

$\sum G_i(V)/R_i(V)$ is a graded commutative ring. Finally, let $F: V \to W$ be a correspondence between two non-singular projective varieties, given by a cycle Z on $V \times W$. Then the correspondence $X \to F^{-1}(X) = \mathrm{pr}_V((V \times X) \cdot Z)$ defines a module-homomorphism of $\sum G_i(W)/R_i(W)$ into $\sum G_i(V)/R_i(V)$. If F is a rational mapping, everywhere defined on V, then the mapping F^* thus defined is a graded-ring-homomorphism. Moreover, if G is an everywhere defined rational mapping of W into a non-singular projective variety U, $F^*(G^*(\ )) = (G \circ F)^*(\ )$.

{The reader must be warned of a few mistakes in this paper, which can be fixed easily as follows. First, lemma 3 is not correct and it could be replaced by the following lemma. Let V^n be a non-singular projective variety in a projective space P_q, A^a a subvariety of V, (B_i), (B_j') two finite sets of subvarieties of V such that the $A \cdot B_i$ are defined. Then there is a set $(C_1, \cdots, C_m)$ of projecting cones in P_q, projecting subvarieties of V from linear spaces of dimension $q-n-1$, such that $\sum (-1)^{t-1} C_t \cdot V \pm X = A$, $X > 0$ and that the $C_t \cdot B_i$, $X \cdot B_i$, $X \cdot B_j'$ are defined. This can be proved by repeated applications of methods of projecting cones. Also the remark (a) in 1 is not correct. But going back to the original form of the associativity of intersection multiplicities [cf. A. Weil, Foundations of algebraic geometry, Amer. Math. Soc. Colloq. Publ., vol. 29, New York, 1946; MR 9, 303; Chap. VI, Th. 5], difficulties can be avoided easily.}

T. Matsusaka (Evanston, Ill.)

2344:

Zariski, Oscar. On the purity of the branch locus of algebraic functions. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 791–796.

Let V^*/k^* be a normalization in K^* of an absolutely irreducible r-dimensional normal algebraic variety V/k, where K^* is a finite separable algebraic extension of the function field K of V and k^* is the algebraic closure of k in K^*. Let P^* be an arbitrary point of V^* and let P be the corresponding point of V. Let $\mathfrak{o}$ and $\mathfrak{o}^*$ be the local rings of P and P^* on V and V^*, respectively, and let $\mathfrak{m}$ and $\mathfrak{m}^*$ be the maximal ideals of $\mathfrak{o}$ and $\mathfrak{o}^*$, respectively. The point P^* is said to be unramified relative to V if: (a) $\mathfrak{o}^*\mathfrak{m} = \mathfrak{m}^*$, and (b) $k^*(P^*)$ is a separable extension of $k(P)$. Let $R = k[x_1, \cdots, x_n]$ be the coordinate ring of an affine part V_a of V containing the point P and let $R^* = k^*[x_1, \cdots, x_n, x_{n+1}, \cdots, x_m]$ be the integral closure of R in K^*. Let $\{g_j(X_i)\}$, $j = 1, \cdots, N$; $i = 1, \cdots, m$, be a basis of the ideal corresponding to P^*, where the X_i, $i = 1, \cdots, m$, are indeterminates. The author proves: (i) The point P^* is unramified if and only if the jacobian matrix of $g_1, \cdots, g_N$ with respect to $X_{n+1}, \cdots, X_m$ has rank $m-n$ at P^*. (ii) (Main theorem) Let Δ be the branch locus of V^* with respect to V and let P be a simple point of V/k. If k is either of characteristic zero or is a perfect field of characteristic $p \neq 0$, then Δ is locally, at P^*, pure $(r-1)$-dimensional. (iii) P^* is unramified if and only if the vector space of local derivations at P^* is obtained from the vector space of local derivations at P by the extension $k(P) \to k^*(P^*)$ of the field of scalars.

The proof, ad absurdum, of (ii) is grounded on the following propositions. (A) Let $x_1, \cdots, x_r$ be a set of uniformizing parameters of the rational point P. Then it is verified that the r partial derivations $\partial/\partial x_i$ of K^*/k', where k' is the algebraic closure of k, map $\mathfrak{o}^*$ into itself. (B) Let k be either of characteristic zero or a perfect field of characteristic p. If the partial derivations $\partial/\partial x_i$, $i = 1, \cdots, r$, map $\mathfrak{o}^*$ into itself, the point P^* is unramified with respect to V. For the proofs of both propositions (A) and (B), the assumption that P is normal is used.

P. Abellanas (Madrid)

2345:

Nagata, Masayoshi. Remarks on a paper of Zariski on the purity of branch loci. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 796–799.

The author proves the two following propositions: (i) Let P be a spot [see: Nagata, Amer. J. Math. **78** (1956), 78–116; MR **18**, 600; p. 86] over a field k of characteristic zero and let r be the dimension of the function field L of P. If there exists a transcendental basis $x_1, \cdots, x_r$ of P over k such that the partial derivations $\partial/\partial x_i$, $i = 1, \cdots, r$, can be extended to integral derivations of P, then P is a regular local ring, and if $\mathfrak{p}$ is the intersection of the maximal ideal $\mathfrak{m}$ of P with $k[x_1, \cdots, x_r]$, then P is unramified over $k[x_1, \cdots, x_r]_{\mathfrak{p}}$. (ii) The main theorem (ii) of Zariski [see the above review] is also true for any field of characteristic $p \neq 0$.

The proposition (i) is a partial generalization of the proposition (B) of Zariski [see the above review] in the sense that (i) does not employ the condition of normality of P when k is of characteristic zero. The proof of (ii) is grounded on the following two propositions. (A′) If $\mathfrak{o}'$ is a ring of quotients of a finite separable integral extension of a normal Noetherian ring $\mathfrak{o}$ and if every prime ideal of rank 1 in $\mathfrak{o}$ is unramified in $\mathfrak{o}'$, then every integral derivation of $\mathfrak{o}$ can be extended to an integral derivation of $\mathfrak{o}'$. (B′) Assume that P is the power series ring over a field k in indeterminates $x_1, \cdots, x_n$; if Q is a normal local ring which is a ring of quotients of a finite separable extension of P, and if $a \in k$, $a^{1/p} \notin k$, let P' be $P[a^{1/p}]$ and let Q' be the derived normal ring of $Q[a^{1/p}]$. Then every prime ideal of rank 1 in P' is unramified in Q'. Furthermore, Q is unramified over P if and only if Q' is unramified over P'. The proposition (B') permits the reduction of the proof of (ii) to the case in which k is the smallest perfect field containing the original k. The proposition (A′) is the one analogous to the proposition (A) by Zariski in the case of a power series ring over a perfect field.

P. Abellanas (Madrid)

2346:

Abellanas, Pedro. Arithmetic-geometric theory of algebraic surfaces. Rev. Mat. Hisp.-Amer. (4) **17** (1957), 65–149. (Spanish)

This paper deals with the theory of algebraic functions of two variables. Let Σ be a field of algebraic functions of two variables over K and let P be a model for Σ. A set S of models for Σ is defined by considering the so-called anti-projections of P, namely, the models of Σ whose coordinate rings contain the coordinate ring of P. With the properties obtained from these definitions, the author is able to define intersection of divisors. Some properties of linear systems of cycles and divisors are obtained (§ 4, 5). In § 6 the differentials are introduced with the assumption that the ground field K is of characteristic zero. This restriction is maintained throughout the rest of the paper. § 7 deals with irregularity of Σ, geometric genus of Σ, genus of a class and linear genus. In the last two paragraphs, using the Joung function, the arithmetic genus is defined and a proof of the Riemann-Roch theorem is given.

E. Lluis (Mexico, D.F.)

2347:

Nakai, Yoshikazu. On the characteristic linear systems of algebraic families. Illinois J. Math. **1** (1957), 552–561.

Sufficient conditions for the completeness of the

characteristic series of maximal algebraic families on varieties defined over arbitrary ground fields are investigated. The main result asserts that if Σ is a maximal family of curves on a non-singular surface V such that $p_g(V)=0$, and if the generic curve C of Σ satisfies the conditions (1) C is non-singular and (2) the cohomological dimension dim $H^1(V,\mathfrak{L}(K-C))$ is zero, then the characteristic series of Σ is complete. Here K denotes a canonical divison and $\mathfrak{L}(K-C)$ is the sheaf of germs of rational functions on V associated with the divison $K-C$. A further result of considerable interest gives the integer $(C^2)-p(C)+1$ as a lower bound for the dimension of the maximal algebraic family on a nonsingular surface that contains a given nonsingular curve C. Here (C^2) and $p(C)$ denote as usual the degree and arithmetic genus of C. *H. T. Muhly* (Iowa City, Iowa)

2348:

Lluis, Emilio; and Samuel, Pierre. On algebraic subgroups of vector spaces. Bol. Soc. Mat. Mexicana **2** (1957), 57–62. (Spanish)

Let K be a perfect field of characteristic $p\neq 0$. A p-polynomial over K is a polynomial whose terms are only p^i-powers of the variables. The authors prove: "If G is an algebraic additive subgroup of the vector space K^n, then G is a rational variety which admits a parametric representation by means of p-polynomials."

P. Abellanas (Madrid)

2349:

Nishimura, Hajime. Some remarks on rational points. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 189–192.

The following theorem is proved. Let U, V be abstract varieties, V complete, let Π be a rational function defined on U with values in V, and let k be a field of definition for U, V, Π. If U has a rational point P over k that is simple on U, then V also has a rational point over k. An example is given to show that the hypothesis, "P is simple", cannot be replaced by the weaker hypothesis, "U is normal at P". *H. T. Muhly* (Iowa City, Iowa)

2350:

Godeaux, Lucien. Involuzioni cicliche appartenenti a superficie algebriche. Matematiche, Catania **11** (1956), 105–106 (1957).

LINEAR ALGEBRA

See also 2386, 2428, 2737a-c.

2351:

✱Cherubino, Salvatore. Calcolo delle matrici. Consiglio Nazionale delle Ricerche: Monografie Matematiche, IV. Edizioni Cremonese, Roma, 1957. vii+322 pp. 4000 Lire.

Durch die vorliegende Monographie des Matrizenkalküls wird die italienische Literatur durch ein, mit den auf diesem Gebiet führenden Werken (Gantmacher, MacDuffee, Schwerdtfeger, Wedderburn) gleichwertiges Werk bereichert. Von den ersten Begriffen des Matrizenkalküls ausgehend werden in derselben unter Anwendung einer wohldurchdachten, übersichtlichen und raumsparenden Bezeichnungsweise klassische und neuere Theorien des Matrizenkalküls entwickelt, wobei einige Abschnitte recht eingehend bearbeitet sind. Dies betrifft insbesondere die Eigenschaften der charakteristischen Gleichungen und Wurzeln, die diesbezüglichen Ausführungen im Fall von ausgezeichneten Matrizentypen (symmetrische, hermitesche u. andere spezielle Matrizen) sowie die Betrachtungen über Funktionen von Matrizen. Das Buch enthält eine übersichtliche Ausführung über die für die Matrizentheorie wichtigen Begriffe aus der abstrakten Algebra (Schiefringe, Schiefkörper, isomorphe Abbildungen, usw.) im Zusammenhang mit ihrer Realisation durch Matrizen. Ein grosser Wert wird auch numerischen Methoden beigelegt, wobei numerische Berechnungen der charakteristischen Wurzeln eingehend besprochen und an Beispielen illustriert werden. Das Buch gliedert sich in vier Kapitel. Das erste Kapitel ist formalen Eigenschaften von Matrizen gewidmet und enthält u.a. den Sylvester-Hadamardschen Satz über den grössten Absolutbetrag einer Determinante, Betrachtungen über quadratische und hermitesche Formen sowie die Differential- und Integralrechnung für Matrizen, deren Elemente als Funktionen einer reellen Veränderlichen gegeben sind. Das 2. Kap. behandelt Struktureigenschaften der Matrizen, insbesondere die mit der Elementarteilertheorie zusammenhängenden Situationen. Das 3. Kap. behandelt Fragen über numerische Berechnungen von charakteristischen Wurzeln. Neben klassischen Resultaten von Sylvester und Bernoulli werden auch neuere Arbeiten von Müller, Ostrowski und Taussky berücksichtigt. Das 4. Kap. ist Theorien über Funktionen von Matrizen gewidmet. {Der Ref. erlaubt sich bei dieser Gelegenheit das ältere Buch von P. Muth, Theorie und Anwendung der Elementartheiler [Teubner, Leipzig, 1899], sowie die anscheinend wenig bekannte und doch schöne und für Anwendungen fruchtbare Theorie von Eduard Weyr [Monatsh. Math. Phys. **1** (1890), 163–200, 201–236] in Erinnerung zu bringen.} *O. Borůvka* (Brno)

2352:

Marcus, M.; Moyls, B. N.; and Westwick, R. Some extreme value results for indefinite Hermitian matrices. II. Illinois J. Math. **2** (1958), 408–414.

Given $a_1\geqq a_2\geqq\cdots\geqq a_n$ and $1\leqq k\leqq n$, let $T_k(a_1, \cdots, a_n)$ denote the set of all real k-tuples $(b_1, \cdots, b_k)$ satisfying

$$\sum_{j=1}^{h} a_j \geqq \sum_{j=1}^{h} b_{i_j} \geqq \sum_{j=1}^{h} a_{n-j+1}$$

for all $1\leqq h\leqq k$ and $1\leqq i_1<\cdots<i_h\leqq k$. Let $E_r(b_1, \cdots, b_k)$ be the rth elementary symmetric function of $b_1, \cdots, b_k$. The authors investigate extreme values of $E_r(b_1, \cdots, b_k)$ as $(b_1, \cdots, b_k)$ varies in $T_k(a_1, \cdots, a_n)$ where $1\leqq r\leqq k\leqq n$. If v is such an extreme value, it is shown that the function E_r takes on the value v at a point $(b_1, \cdots, b_k)\in T_k(a_1, \cdots, a_n)$ such that $(b_1, \cdots, b_k)$ is contained in some set $T_k(a_{i_1}, \cdots, a_{i_k})$, where $a_{i_1}, \cdots, a_{i_k}$ are the first s and last $k-s$ of the a's $(0\leqq s\leqq k)$. Also, an expression of v in terms of this reduced set of k a's is given. This result is then applied to a discussion of the maximum and minimum of $E_r((Hx_1, x_1), \cdots, (Hx_k, x_k))$, where H is a Hermitian (in general, indefinite) transformation in the unitary n-space U^n, $1\leqq r\leqq k\leqq n$, and where the maximum and minimum are taken over all sets of k orthonormal vectors $x_1, \cdots, x_k$ in U^n. The following result is also proved. If $a_1\geqq a_2\geqq\cdots\geqq a_m$ $(1\leqq m\leqq n)$ are m of the n eigenvalues of H and if $(b_1, \cdots, b_k)$ is in $T_k(a_1, \cdots, a_m)$ $(1\leqq k\leqq m)$, then there exist k orthonormal vectors $x_1, \cdots, x_k$ in U^n such that $(Hx_i, x_i)=b_i$ $(1\leqq i\leqq k)$.

Ky Fan (Notre Dame, Ind.)

2353:

Marcus, Marvin. On subdeterminants of doubly stochastic matrices. Illinois J. Math. 1 (1957), 583–590.

Let A be a complex matrix of order n, and $d_r(A)$ a typical subdeterminant of order r; summations are extended over all subdeterminants of the appropriate order. It is proved that if A is of rank k, then for $1<r\leq k$ the inequality $\sum |d_r(A)|^2\leq\binom{k}{r}k^{-r}(\sum |d_1(A)|^2)^r$ holds; cases of equality are discussed. This generalises a result of H. Richter [Arch. Math. 5 (1954), 447–448; MR 16, 106]. Specialising to the case where A is doubly stochastic, the inequality $\sum d_r^2(A)\leq\binom{k}{r}$ is obtained, again with a full discussion of equality. Finally, the question of how many subdeterminants of order r of a doubly stochastic matrix can have absolute value 1 is raised and answered.

J. H. Williamson (Cambridge, England)

2354:

Fan, Ky. Topological proofs for certain theorems on matrices with non-negative elements. Monatsh. Math. 62 (1958), 219–237.

Some of the algebraic results of this paper seem to be new, but, as the title implies, the interest lies mainly in the methods of proof and the extraction of the topological content of the theorems. Since essentially all results are shown to flow from the matrix-free Theorem 1, it will be quoted in full.

"Theorem 1. Let S^n be an n-simplex and let S_i^{n-1} $(0\leq i\leq n)$ be the $(n-1)$-dimensional faces of S^n. Let $f_i(0\leq i\leq n)$ be $n+1$ real-valued continuous functions defined on S^n such that the following two conditions are satisfied. (a) For any two indices i, j, we have $f_i(x)\leq f_j(x)$ for $x\in S_i^{n-1}$. (b) For any ordered pair of points x, y of S^n, there is at least one index i for which $f_i(x)\leq f_i(y)$. Then (i) there exists a point $\hat{x}\in S^n$ such that (1) $f_0(\hat{x})=f_1(\hat{x})=\cdots=f_n(\hat{x})$; (ii) the equality

$$(2)\qquad \operatorname*{Min}_{x\in S^n}\ \operatorname*{Max}_{0\leq i\leq n} f_i(x)=\operatorname*{Max}_{x\in S^n}\ \operatorname*{Min}_{0\leq i\leq n} f_i(x)$$

holds; (iii) for any point $\hat{x}\in S^n$ satisfying (1), the common value of $f_i(\hat{x})$ $(0\leq i\leq n)$ is equal to the common value in (2)."

This is shown to imply, as a special case, the classical Frobenius theorem that, if A is non-negative and indecomposable, it has a positive eigenvector x belonging to a positive eigenvalue λ, and λ is maximal among all eigenvalues of A. Thereafter, the paper takes up the positivity of $(\rho I-A)^{-1}$ and of principal minors of $\rho I-A$, with conditions equivalent to these, and the comparison of the maximal eigenvalue of A with those of its principal minors of order $n-1$; it concludes with inequalities of Price giving bounds on determinants with dominant diagonals.

A. S. Householder (Oak Ridge, Tenn.)

2355:

Householder, A. S. On matrices with non-negative elements. Monatsh. Math. 62 (1958), 238–242.

In a paper by the reviewer [reviewed above], certain known and some new results concerning matrices with non-negative elements were proved by using elementary topological facts. The present paper supplies purely algebraic proofs for some of the theorems discussed in the reviewer's paper, with one result sharpened. Let A be an n-square matrix with non-negative elements, and let λ be its maximal eigenvalue. Let λ_i denote the maximal eigenvalue of the submatrix obtained from A by deleting its ith row and ith column. If $\lambda>\rho>\max_{1\leq i\leq n}\lambda_i$, it was shown by the reviewer that $(A-\rho I)^{-1}$ exists (I being the identity matrix) and all its elements are positive. In the present paper, the author proves that, if A is irreducible and if $\lambda>\rho>\lambda_i$ for some i, then $(A-\rho I)^{-1}$ exists and its elements on the ith row and ith column are positive.

Ky Fan (Notre Dame, Ind.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 2319.

2356:

Onodera, Takesi; and Tominaga, Hisao. On strictly Galois extensions of degree p^e over a division ring of characteristic p. Math. J. Okayama Univ. 7 (1957), 77–81.

Let K be a division ring, and let D be a Galois subring corresponding to a non-trivial finite group G of automorphisms of K. In a natural way K and GD may be regarded as GD-modules, where GD denotes the group ring defined by G and D. In a lemma the authors obtain (as a fairly easy consequence of results of F. Kasch's fundamental paper [Math. Ann. 126 (1953), 447–463; MR 15, 597]) the following noteworthy result: If K/D has dimension $[K:D]$ equal to the order of G, then K and GD are isomorphic GD-modules. (If G is outer, this result specializes to a theorem of T. Nakayama [Proc. Imp. Acad. Tokyo 16 (1940), 532–536; MR 2, 344]. Although the authors refer to a paper [Duke Math. J. 21 (1954), 87–105; MR 15, 848] of Amitsur, they do not specifically mention the fact that a theorem in this paper guarantees the existence of the non-outer case of their lemma.) If $u\in K$ is an image of a regular element (=unit) of GD under any module isomorphism of GD and K, then u is called a G-normal basis element (G-n.b.e) of K/D. The major portion of the paper is devoted to the proof of the equivalence of (I) and (II) below: (I) If $\sum_{S\in G} uS\neq 0$, then u is a G-n.b.e. of K/D; (II) K has characteristic $p>0$, and $[K:D]=p^e$. (Note: since G is non-trivial, $K\neq D$.) The method of proof is first to show that (I) is equivalent to: (I′) GD is completely primary (in the sense that the set of non-units of GD form a maximal non-zero ideal of GD). Then (I′)⇔(II) is obtained. If the authors' proofs seem a trifle long, this is mainly because they, in effect, rederive certain known results; e.g., they do not make use of the known equivalence of (I′) and (II) for D a field.

The authors mention in the text that the equivalence (I)⇔(II) had been announced previously by the reviewer [Bull. Amer. Math. Soc. 63 (1957), 95–96] when K is a field. Nevertheless, the present paper constitutes the first published proof of this. A certain more general theorem announced in the abstract cited has been extended to simple rings by the reviewer [#2357 below], and the extension to these rings of the results of the paper under review is obtained as a corollary.

C. C. Faith (University Park, Pa.)

2357:

Faith, Carl C. Galois extensions in which every element with regular trace is a normal basis element. Proc. Amer. Math. Soc. 9 (1958), 222–229.

Let $\mathfrak{K}\supset\Delta\supseteq\mathfrak{F}$ be simple rings, $\mathfrak{G}$ the Galois group of $\mathfrak{K}/\mathfrak{F}$, $\mathfrak{P}$ the Galois group of $\Delta/\mathfrak{F}$, and $\mathfrak{H}=\{H_i\}$ the normal subgroup corresponding to Δ. Put $t_{\mathfrak{H}}(w)=\sum H_i(w)$, and assume $(\mathfrak{K}:\mathfrak{F})=$order $\mathfrak{G}$, $(\mathfrak{K}:\Delta)=$order of $\mathfrak{H}$. An element w

generates a $\mathfrak{G}$-normal basis of $\mathfrak{K}/\mathfrak{F}$ if the set $\{G_i(w), G_i \in \mathfrak{G}\}$ is an $\mathfrak{F}$-basis of $\mathfrak{K}$. The main result is that the conditions $\chi(\mathfrak{F})=p$, order of $\mathfrak{H}=p^e$ are necessary and sufficient conditions that the following hold: an element w generates a $\mathfrak{G}$-normal basis of $\mathfrak{K}/\mathfrak{F}$ if and only if the trace $t_{\mathfrak{H}}(w)$ generates a $\mathfrak{P}$-normal base of $\Delta/\mathfrak{F}$. For $\Delta=\mathfrak{F}$, this means that $t_{\mathfrak{G}}(w)$ is a regular element in $\mathfrak{F}$. The main tool of the proof is the operator isomorphism between $\mathfrak{K}$ and $\mathfrak{F}[\mathfrak{G}]$ and the determining of conditions that the ideal generated by the elements $\{H_i-1, H_i \in \mathfrak{H}\}$ belong to the radical of $\mathfrak{F}[\mathfrak{G}]$. Two conditions are given for the existence of a $\mathfrak{G}$-normal base: (1) $(\mathfrak{K}:\mathfrak{F})$=order of $\mathfrak{G}$; (2) the centralizer of $\mathfrak{F}$ in $\mathfrak{K}$ is simple. The author has pointed out to the reviewer that the following condition should be added: (3) The ring generated by all regular elements effecting inner automorphisms of $\mathfrak{K}$ belonging to $\mathfrak{G}$ is simple.

S. A. Amitsur (Notre Dame, Ind.)

2358:

Reiner, Irving. A theorem on continued fractions. Proc. Amer. Math. Soc. 8 (1957), 1111–1113.

Let K be a sfield. Let $R=K[x]$ be the ring of polynomials in an indeterminate x with coefficients in K, where we assume that x commutes with all elements of K. For $f_1, \cdots, f_n \in R$, we define $[f_1, \cdots, f_n] \sim P/Q$ by the relation

$$\begin{pmatrix} f_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} f_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} f_n & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} P \\ Q \end{pmatrix};$$

or, more precisely,

$$PQ^{-1}=f_1+\frac{1}{f_2}+\frac{1}{f_3}+\cdots+\frac{1}{f_n}.$$

Let $f\to f^*$ denote any homomorphism of $(R, +)$ into itself, which leaves K elementwise fixed, and satisfies $(af)^*=af^*$ for all $a \in K$, $f \in R$. Then $[f_1, \cdots, f_k] \sim P/Q$, $P, Q \in K$, implies $[f_1^*, \cdots, f_k^*] \sim P/Q$. The novelty of the result is that the homomorphism is an additive one and independent of the multiplicative property of the sfield.

L. K. Hua (Peking)

2359:

Lesieur, Léonce; et Croisot, Robert. La notion de résiduel essentiel. C. R. Acad. Sci. Paris 246 (1958), 357–360.

Let A be a not necessarily commutative ring with ascending chain condition for left ideals, and let L be the lattice of left ideals in A. If X, Y are elements of L, the left residual $X^{\cdot}.Y$ is the maximal two-sided ideal B such that $BY\subseteq X$. A two-sided ideal T is called an essential residual of X if there exists an element Y containing X such that $T=X^{\cdot}.Y$, and if $X\subset Z\subseteq Y$ implies $X^{\cdot}.Z=X^{\cdot}.Y$. Every maximal proper left residual of X is an essential residual of X and every essential residual of X is prime. Several properties of essential residuals are derived with a view to the application made of this concept in the note reviewed below.

D. C. Murdoch (Vancouver, B.C.)

2360:

Lesieur, Léonce; et Croisot, Robert. Une propriété caractéristique des idéaux tertiaires. C. R. Acad. Sci. Paris 246 (1958), 517–520.

In a previous note [C. R. Acad. Sci. Paris 243 (1956), 1988–1991; MR 18, 637], the authors have developed a decomposition theory for ideals in a noncommutative ring with ascending chain condition, according to which every ideal is a reduced intersection of a finite number of tertiary ideals (see above reference for definition). In such a decomposition, the number of tertiary components and their associated primes are uniquely determined. It is here shown that a proper left ideal is tertiary if and only if it admits a unique essential residual. Under the assumption that the lattice of left ideals is semi-modular, it is shown that the essential residuals of a left ideal X coincide with the prime ideals associated with the tertiary components in a reduced decomposition of X.

D. C. Murdoch (Vancouver, B.C.)

2361:

Posner, Edward C. Derivations in prime rings. Proc. Amer. Math. Soc. 8 (1957), 1093–1100.

Let A be a prime ring and let d, d_1, d_2 be derivations of A. It is shown that: 1) if d_1d_2 is also a derivation of A and the characteristic of $A\neq 2$, then either d_1 or d_2 is zero; 2) if $ad(a)-d(a)a$ belongs to the centre of A for all $a \in A$, then $d=0$ or A is commutative. The proof of the last result is carried out separately for characteristics 2 and 3 and for rings of higher characteristics.

S. A. Amitsur (Notre Dame, Ind.)

2362:

Herstein, I. N. Jordan derivations of prime rings. Proc. Amer. Math. Soc. 8 (1957), 1104–1110.

A Jordan derivation of an associative ring A is a linear mapping: $a\to a'$ of A into itself satisfying: 1) $(a^2)'=a'a+aa'$, 2) $(aba)'=a'ba+ab'a+aba'$. Requirement (2) follows from (1) if the characteristic of $A\neq 2$. The result of the paper is that if A is a prime ring which is not an integral domain of characteristic 2, then any Jordan derivation of A is an ordinary derivation, i.e., it satisfies: $(ab)'=a'b+ab'$.

S. A. Amitsur (Notre Dame, Ind.)

2363:

Herstein, I. N. Certain submodules of simple rings with involution. Duke Math. J. 24 (1957), 357–364.

Recent studies by the author [Amer. J. Math. 78 (1956), 629–649; MR 18, 714] and W. E. Baxter [reviewed below] concerning Lie and Jordan subsystems in simple associative rings with involution are continued here. Let A be a simple ring with involution, K and S the sets of skew and symmetric elements respectively, Z the center of A. Assume that A has characteristic $\neq 2$ (and also $\neq 3$ for the second result below), and either that $Z=0$ or that A is more than 16-dimensional over Z. It is shown that (1) if U is any additive subgroup of K such that $U\circ S\subset U$ (where $u\circ s=us+su$ is the Jordan product in A), then either $U=0$ or $U=K$; and (2) if U is any additive subgroup of S such that $[U, [K, K]]\subset U$, then either $U\subset Z$ or $U\supset[K, S]$.

R. D. Schafer (Storrs, Conn.)

2364:

Baxter, Willard E. Lie simplicity of a special class of associative rings. II. Trans. Amer. Math. Soc. 87 (1958), 63–75.

[For Part I, see Proc. Amer. Math. Soc. 7 (1956), 855–863; MR 18, 557.] Let A be a simple associative ring with an involution defined on it (an anti-automorphism $a\to a^*$ of period two). Let K be the set of elements $x \in A$ satisfying $x^*=-x$. Then K is a Lie subring of A under Lie multiplication $[x, y]=xy-yx$. An additive subgroup U of K is called a Lie ideal of K if $uk-ku \in U$ for all $u \in U$ and $k \in K$. In this paper the author considers the Lie structure of the Lie ideal $[K, K]$, the additive subgroup of K generated by all elements of the form $kl-lk$, $k, l \in K$. Let Z be the centre of A. If $x^*=x$ for all x in Z, the involution is of the first kind. If $x^*=-x$ for at least one non-zero element of Z, the involution is of the second kind. (An involution not of the first kind is obviously of the second kind.) The

main result is: Let A be of characteristic $\neq 2$ with $Z=(0)$ or with A at least 16-dimensional over Z. Then (i) $[K, K]$ is a simple Lie ring, if the involution is of the first kind, and (ii) $[K, K]$ modulo its intersection with Z is a simple Lie ring, if the involution is of the second kind.

K. G. Wolfson (New Brunswick, N.J.)

2365:

Pham, Daniel Tinh-Quat. Sur les anneaux indexables. C. R. Acad. Sci. Paris **245** (1957), 1683–1685.

Let R be an associative ring with an identity. Consider the equivalence classes $\{x\}$ determined by the equivalence relation $x \sim y$ if and only if $x=\alpha y\beta$ (where α, β are units of R). Then R is indexable if it is possible to linearly order the classes (in at least one way) by a relation $\leqq$ satisfying: (1) $\{0\}\leqq\{x\}$ if $x\neq 0$; (2) $\{xy\}\leqq\{x\}$; $\{xy\}\leqq\{y\}$; (3) between $\{0\}$ and $\{x\}$ (including $\{0\}$ but not $\{x\}$) there are a finite number n of classes different from the class of units.

The class $\{x\}$ is then of index n. A class of index one is called minimal. The two-sided ideal generated by the elements of a minimal class is called a kernel. R is of finite index if the total number of classes is finite. The ring is uninodular if it contains a unique minimal class. The ring of $m\times m$ matrices over a division ring and the ring of integers modulo m are indexable rings of finite index.

This paper contains the statements of two decomposition theorems (without proofs). The first states that an indexable ring of finite index with idempotent kernels (a kernel N satisfies either $N^2=N$ or $N^2=0$) is a direct sum of uninodular rings if and only if the two-sided annihilator of each kernel is a principal right ideal and a principal left ideal. The second gives necessary and sufficient conditions for the decomposition of a ring of finite index (with arbitrary kernels) into a direct sum of indecomposable indexable rings.

K. G. Wolfson (New Brunswick, N.J.)

2366:

Johnson, R. E. Rings with unique addition. Proc. Amer. Math. Soc. **9** (1958), 57–61.

A ring $\{R; +, \cdot\}$ is said to have a unique addition if there exists no other ring $\{R; +', \cdot\}$ having the same multiplicative semi-group $\{R; \cdot\}$. In the present paper the author extends some results of the reviewer [Bull. Amer. Math. Soc. **54** (1948), 758–764; MR **10**, 96], concerned with uniqueness of addition for certain semi-simple rings, to a larger class of rings. For the statement of these results, some definitions are needed. For any ring R, denote by $\mathscr{L}(R)$ the lattice of all right ideals of R and by $\mathscr{L}^\blacktriangle(R)$ the sublattice of all $A\in\mathscr{L}(R)$ such that $A\cap B\neq 0$ for every nonzero $B\in\mathscr{L}(R)$. For a set $A\subset R$, let A^r denote the right annihilator of A in R and, for $x\in R$, let $x^{-1}A$ denote the set $\{y: xy\in A\}$. Set $R^\blacktriangle=\{x: x^r\in\mathscr{L}^\blacktriangle(R)\}$; $A^s=\{x: x^{-1}A\in\mathscr{L}^\blacktriangle(R)\}$; $\mathscr{L}^s(R)=\{A^s: A\in\mathscr{L}(R)\}$. Then s is a closure operation on $\mathscr{L}(R)$. A minimal nonzero element of $\mathscr{L}^s(R)$ is called an "atom" and s is called "atomic" if every nonzero element of $\mathscr{L}^s(R)$ contains an atom. The union of all atoms of $\mathscr{L}^s(R)$ is called the "base" of R. Now, let R be any ring with $R^\blacktriangle=0$ for which s is atomic. Then R will have a unique addition in each of the following instances: (1) The base of R has a zero right annihilator and $\mathscr{L}^s(R)$ has no isolated atoms; (2) the right annihilator of each atom in $\mathscr{L}^s(R)$ is not maximal in $\mathscr{L}^s(R)$. Examples show that these conditions cannot, in general, be dropped.

C. E. Rickart (New Haven, Conn.)

2367:

Fujisaki, Genjiro. On the zeta-function of the simple algebra over the field of rational numbers. J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1958), 567–604.

Let A be a simple algebra of finite rank over the rational field. It is known (Hey, Zorn, Leptin) that the zeta-function of the algebra A satisfies a functional equation and can be continued to a meromorphic function, having a finite number of poles, on the entire complex plane. In the present paper, the author proves these results by the method of invariant integrals over locally compact topological groups which was introduced by J. Tate and the reviewer to obtain similar results for the zeta-function of an algebraic number field. The author first considers local algebras A_p and local zeta-functions of such algebras. Then he defines the notion of valuation vectors and idèles for the algebra A and studies their algebraic and topological properties. Finally, the author defines the zeta-function of A as an integral over the idèle group of A and proves the main properties of the function, in essentially the same way used in the case of number fields.

K. Iwasawa (Cambridge, Mass.)

2368:

Iséki, Kiyoshi. Ideals in semirings. Proc. Japan Acad. 34 (1958), 29–31.

This paper is a continuation of an earlier one by the author on prime ideals in semirings [Proc. Japan Acad. **32** (1956), 554–559; MR **19**, 10]. An ideal P of a semiring S is called prime if $AB\subset P$, A and B ideals of S, implies $A\subset P$ or $B\subset P$, completely prime if $ab\in P$, a and b in S, implies $a\in P$ or $b\in P$, and completely semiprime if $a^2\in P$ implies $a\in P$. The author proves that a prime ideal P is completely prime (i) if and only if $ab\in P$ implies $ba\in P$; (ii) if and only if P is completely semiprime. He also proves that a semiring S is regular if and only if $R\cap L=RL$ for any right ideal R and left ideal L of S.

W. E. Deskins (East Lansing, Mich.)

2369:

Iséki, Kiyoshi. Quasi-ideals in semirings without zero. Proc. Japan Acad. **34** (1958), 79–81.

A subsemiring A of a semiring S without zero is called a quasiideal if $AS\cap SA\subseteq A$. The author proves that the intersection of a minimal right and a minimal left ideal of S is a minimal quasiideal and, conversely, that every minimal quasiideal can be so obtained. From this he concludes that a minimal quasiideal Q is a division semiring, that it contains an idempotent e such that $Q=eSe$, and that it is isomorphic with any other minimal quasiideal of S. O. Steinfeld has obtained similar results for quasiideals in semigroups and rings [Publ. Math. Debrecen **4** (1956), 262–275; Acta Sci. Math. Szeged **17** (1956), 170–180; MR **18**, 790, 637].

W. E. Deskins (East Lansing, Mich.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

See 2363, 2364.

HOMOLOGICAL ALGEBRA

See 2702.

GROUPS AND GENERALIZATIONS

See also 2319, 2484, 2992.

2370:

Adyan, S. I. Unsolvability of some algorithmic problems in the theory of groups. Trudy Moskov. Mat. Obšč. 6 (1957), 231–298. (Russian)

The paper leans heavily on the method which P. S. Novikov has developed to prove the algorithmic insolubility of the word problem [Trudy Mat. Inst. Steklov 1955, no. 44; MR **17**, 706], conjugacy problem [Izv. Akad. Nauk SSSR. Ser. Mat. **18** (1954), 485–524; MR **17**, 706] and isomorphism problem (unpublished) for groups. The author proves the algorithmic insolubility of a number of further group-theoretical problems. A property α of a group F_0 shall be called intrinsic if every group isomorphic to F_0 also has the property α. The discrimination problem for an intrinsic property α is the problem of finding an algorithm to decide for every finitely presented group F whether α holds in it or not. An intrinsic property α is called special if it does not hold in any group with an insoluble word problem. An intrinsic property β of a group F_0 shall be called hereditary if every group that is isomorphic to a subgroup of F_0 also has the property β. A hereditary property β is called non-trivial if the free group on two generators does not have the property β.

Theorem 1: Let F_0 be a fixed finitely presented group. There is no algorithm to decide for every finitely presented group F whether it is isomorphic to F_0 or not. Theorem 2: Let α be an arbitrary intrinsic property, β a non-trivial hereditary property. If there exists at least one finitely presented group in which the logical union (α and β) holds, then the discrimination problem for the property (α and β) is algorithmically insoluble. Examples of non-trivial hereditary properties are those of being finite, periodic, abelian, nilpotent, or soluble. It follows, in particular, that there is no algorithm to decide for every finitely presented group whether it is finite of a given order k or not, or whether it is nilpotent of a given class k or not. Theorem 3: Let α be a special intrinsic property. If there exists at least one finitely presented group F_0 in which α holds, then the discrimination problem for α is insoluble. This theorem has the following consequence: not only is the word problem for groups insoluble, but so is the "meta-problem" of setting up an algorithm to decide whether a given finitely presented group has an insoluble word problem or not. Theorem 4: The discrimination problem is insoluble for the properties of (i) containing a free subgroup of given finite rank k, (ii) being simple. These four theorems are the outcome of a long and detailed analysis of transformations in Novikov's centrally-symmetric group [loc. cit.] and in similarly constructed groups. The combinatorial details are formidable.

K. A. Hirsch (London)

2371:

Adyan, S. I. Finitely presented groups and algorithms. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 9–12. (Russian)

The author proves the following theorem: Let α be an intrinsic group property [see the preceding review]. If there exists a finitely presented group F_1 with the property α, as well as a finitely presented group F_2 that cannot be embedded in any finitely presented group with the property α, then there is no algorithm to decide for an arbitrary finitely presented group F whether or not it has the property α. This theorem extends previous results of the author [see the preceding review] and is a group-theoretical analogue to a theorem of Markov [Trudy. Mat. Inst. Steklov. **1954**, no. 42; MR **17**, 1038] on associative systems. The embedding condition on F_2 cannot be replaced by the mere condition that F_2 should not have the property α. For example, finitely presented groups may or may not coincide with their derived group. Nevertheless there is an algorithm to decide for any finitely presented group G whether or not it coincides with its derived group (namely, whether the abelian group G/G' is trivial or not). A review of further results announced in the note is postponed until the detailed proofs are available.

K. A. Hirsch (London)

2372:

Sadovskiĭ, L. E. Structure isomorphisms of a free metabelian group. Mat. Sb. N.S. **42(84)** (1957), 445–460. (Russian)

Let F_r be a free group of rank r and $\Gamma_3(F_r)$ the third group in its lower central series. Then $F_r/\Gamma_3(F_r)=G_r=G$ is called the free metabelian group of rank r. If $x_1, \cdots, x_r$ are generators of G, then an element g of G has a unique form

$$g=x_1^{e_1}x_2^{e_2}\cdots x_r^{e_r}\prod_{i>j}(x_i, x_j)^{f_{ij}},$$

and if $r\geqq 2$ the derived group G' is the center of G. This paper is concerned with a subgroup lattice isomorphism φ of a free metabelian group G_r with another group G^*. G^* will also be a free metabelian group of rank r, but φ is not necessarily a group isomorphism. G^* has generators $y_1, \cdots, y_r$ such that for the cyclic subgroups of G_r, we have

$$[x_i]\varphi=[y_i] \quad (i=1, \cdots, r), \quad [(x_i,x_j)]\varphi=[(y_i,y_j)],$$

$$[x_1^{e_1}x_2^{e_2}\cdots x_r^{e_r}\prod_{i>j}(x_i, x_j)^{f_{ij}}]\varphi=[y_1^{e_1}\cdots y_r^{e_r}\prod_{i>j}(y_i, y_j)^{f_{ij}}k],$$

where in the third relation $k=\prod (y_i, y_j)^{\lambda e_i e_j}$ and λ is some rational such that all $\lambda e_i e_j$ are integers. The images of the cyclic groups above determine the most general lattice isomorphism between G and G^*.

Marshall Hall, Jr. (Columbus, Ohio)

2373:

Karrass, A.; and Solitar, D. On free products. Proc. Amer. Math. Soc. 9 (1958), 217–221.

Let G be a group which is the free product of finitely many groups A_i, each of which is a finite extension of a free group F_i of finite rank, or is the identity. Such a G is called a free product of finite type. Let F be the kernel of the natural homomorphism of G onto the direct product of the finite groups A_i/F_i. Then it is shown that F is free and, of course, of finite index in G. Primarily, from this fact, it follows that free products of finite type have many of the properties of free groups of finite rank. In particular, a finitely generated subgroup H of G which contains a non-trivial normal subgroup N is necessarily of finite index in G. Also G is hopfian; i.e., if N is a normal subgroup such that $G/N\cong G$, then $N=1$. Furthermore, the intersection of two finitely generated subgroups of G is itself finitely generated.

Marshall Hall, Jr. (Columbus, Ohio)

2374:

Fox, Ralph H. Free differential calculus. III. Subgroups. Ann. of Math. (2) **64** (1956), 407–419.

This paper continues the work in group theory begun in parts I and II [Ann. of Math. (2) **57** (1953), 547–560; **59** (1954), 196–210; MR **14**, 843; **15**, 931]. The results are applied to the theory of branched coverings [Fox, Algebraic geometry and topology, pp. 243–257, Princeton

Univ. Press, 1957; p. 243], and, in particular, to the cyclic coverings of knots.

A variant of the theorem of Reidemeister and Schreier is established: let F be a subgroup of index ν in a group G, and F^* the free product of F with a free group of rank $\nu-1$; then a presentation W^*/S^* for F^* is obtained from a presentation X/R for G without use of coset representatives. As noted by Kyle, the representation of G by permutation of its cosets modulo F induces an extended monomial representation ω of the group ring JX by ν-by-ν matrices over JW; this representation ω carries a Jacobian matrix for G into one for F^*, from which invariants of F are easily recovered.

This applies directly to the fundamental group F of an unbranched covering $\mathscr{X}$ of a space $\mathscr{X}_0$ with fundamental group G. Now let $\mathscr{X}_0=\mathscr{X}-\mathscr{L}$, $\mathscr{L}$ a subcomplex of a complex $\mathscr{X}$, and let $\mathscr{Y}$ be a branched covering of $\mathscr{X}$ with singular set a subcomplex of $\mathscr{L}$. Under suitable hypotheses, notably that the intersection with $\mathscr{X}_0$ of the open star of each vertex of $\mathscr{L}$ have homotopy type of the circle, the fundamental group of $\mathscr{Y}$ can be obtained explicitly from that of the associated unbranched covering $\mathscr{X}$ of $\mathscr{X}_0$ by introducing additional, branch, relations. For $\mathscr{L}$ a locally flat polyhedral $(q-2)$-manifold in the q-sphere $\mathscr{X}$, a formula is obtained relating the rank of $\pi_1(\mathscr{Y})$ to the cycle structure of the permutation representing a generator of G. Further results concern the one-dimensional homology of the branched, and the Alexander polynomial of the unbranched, cyclic coverings of a knot.

R. C. Lyndon (Ann Arbor, Mich.)

2375:

Ono, Tamio. An analytical proof of the fundamental theorem on finite abelian groups. Proc. Japan Acad. 33 (1957), 587.

Ingénieuse démonstration, consistant à observer que: (1) le nombre des caractères d'un groupe abélien fini G est fini; (2) pour tout $t\in G$ tel que $t\neq e$, il existe un caractère χ tel que $\chi(t)\neq 1$, en considérant dans l'algèbre de groupe la translation par t, qui a une valeur propre distincte de 1. Désignant par χ_k les caractères, on considère ensuite les sousgroupes G_k "orthogonaux" à tous les χ_i d'indice $i\leq k$, et on voit sans peine que G_k est somme directe de G_{k+1} et d'un groupe cyclique.

J. Dieudonné (Evanston, Ill.)

2376:

Altman, Mieczyslaw. Généralisation aux groupes abéliens de la théorie de F. Riesz. C. R. Acad. Sci. Paris **246** (1958), 1135–1138.

In this note it is shown how to replace topological hypotheses by algebraic in order to get much of the Riesz-Schauder theory of completely continuous operators. Let T be an endomorphism of an abelian group X, let T^0 be the identity endomorphism, let $T^{n+1}=T(T^n)$, let $L_n=T^n(X)$, and let G_n be the null space of T^n. T is said to satisfy the ascending (descending) chain condition if the chain $0\subset G_1\subset G_2\subset\cdots$ (the chain $X=L_0\supset L_1\supset L_2\supset\cdots$) has only finitely many distinct elements. If T satisfies both chain conditions, the least p such that $L_n=L_p$ for all $n>p$ is also the least p such that $G_n=G_p$ for all $n>p$; then X is the direct sum of L_p and G_p. If T is not nilpotent, then (Theorem 2) $T=H+K$, where H is an automorphism of X and $K(X)=L_p$.

M. M. Day (Urbana, Ill.)

2377:

Stojaković, Mirko. Sur une relation d'ordre dans le groupe symétrique. Univ. Beogradu. Godišnjak Filozof. Fak. Novom Sadu **1** (1956), 281–292. (Serbo-Croatian. French summary)

The lexicographic arrangement of elements of a symmetric group σ_n is introduced in this paper; thus the structure of the group σ_n is induced on the initial part $1, 2, \cdots, n!$ of the set N of natural numbers. Several theorems are proved concerning the relations between the groups σ_{n-2}, σ_{n-1} and the group σ_n, and it is demonstrated how this relation can be used for the automatic construction of the Cayley table of composition of the group σ_n, assuming that the table of the σ_{n-1} is known.

T. P. Anđelić (Belgrade)

2378:

Nakamura, Kirio. Über die Ordnung gewisser Untergruppen von GL (q, p). Nagoya Math. J. **12** (1957), 191–193.

Let p $(\neq 2)$ and q be distinct primes. If M is an irreducible nilpotent subgroup of $\mathrm{GL}(q, p)$ whose order $|M|$ is odd and prime to p, then $|M|<p^q$.

W. Ledermann (Manchester)

2379:

Reiner, Irving. Normal subgroups of the unimodular group. Illinois J. Math. **2** (1958), 142–144.

Let Γ be the proper unimodular group of 2×2 matrices (with rational integral entries). The author finds a large number of normal subgroups $\Omega(s, p)$ of Γ which have finite index, yet which contain no congruence subgroup $\Gamma(m)=\{A|\ A\equiv I \bmod m\}$. Let f be the natural mapping of $\Gamma(p)$ onto $\Delta(p)=\Gamma(p)/\Gamma'(p)$. Set $\Delta^s(p)=\{X^s|\ X\in\Delta(p)\}$. Then $\Omega(s, p)$ is $f^{-1}\{\Delta^s(p)\}$. Fricke [Math. Ann. **28** (1887), 99–118] and Pick [ibid. 119–124] defined $\Omega(2, p)$.

J. L. Brenner (Palo Alto, Calif.)

2380:

Reiner, Irving. A new type of automorphism of the general linear group over a ring. Ann. of Math. (2) **66** (1957), 461–466.

Let $R=K[x]$ be the ring of polynomials in an indeterminate x over a field K, and K^* the multiplicative group of K. Let $\mathrm{GL}_2(R)$ be the 2×2 matrices invertible over $K[x]$. The group has the following three types of automorphisms: (1) $u\to tut^{-1}$, $t\in\mathrm{GL}_2(R)$; (2) $u\to\lambda(\det u)u$, where λ is a mapping of K^* into itself such that $a\lambda(a^2)=1$ if and only if $a=1$; and (3) $u\to u^\sigma$, where σ is an automorphism of R. In addition, the author introduces a new type of automorphism: Let y_m $(\in R)$ be any set of additive generators of R and $y_0=1$. The mapping

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\leftrightarrow\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \begin{pmatrix}a&0\\0&1\end{pmatrix}\leftrightarrow\begin{pmatrix}a&0\\0&1\end{pmatrix}(a\in K^*), \begin{pmatrix}1&x^m\\0&1\end{pmatrix}\leftrightarrow\begin{pmatrix}1&y_m\\0&1\end{pmatrix}$$

gives an automorphism of $\mathrm{GL}_2(K)$. The author proves that the group of automorphisms of $\mathrm{GL}_2(K)$ is generated by these four types of automorphisms.

L. K. Hua (Peking)

2381:

Kostant, Bertram. A characterization of the classical groups. Duke Math. J. **25** (1958), 107–123.

The classical groups with their usual representations are characterized: Let $\mathfrak{g}^0$ be a complex Lie algebra of linear transformations of an m-dimensional vector space and irreducible. $\mathfrak{g}^0$ is the Lie algebra of all linear transformations if and only if it contains a non-nilpotent element of rank 1. If $\mathfrak{g}^0$ has a bilinear invariant, it is the Lie algebra of all such linear transformations if and only if it contains a non-nilpotent element of rank 2. The second part is also reformulated in a version for real vectorspaces. The theorem is proved by arguments on representation weights; the conditions are essentially equivalent to that

of the existence of a weight μ that is orthogonal to all weights $\nu \neq \pm\mu$.

The results are applied in the tangent space V_o of a point o of a Riemannian manifold. The curvature belonging to the bivector α can be written as $K(\alpha) = \frac{1}{4}(TA, A)/(A, A)$, where A is the skew symmetric tensor that describes α in a natural way and T is an operator on the Lie algebra $\mathfrak{a}$ of skew symmetric tensors. The Cartan identity for the curvature tensor in symmetric manifolds is equivalent to the condition that ad A commutes with T for all A from the range of T, or, in other words, that the range of T is a subalgebra of $\mathfrak{a}$, and the eigenspaces (not of eigenvalue 0) are ideals. The curvature can be stationary for a certain α if and only if α is an eigenvector of T. The question is raised whether non-zero eigenvalues are possible. The bivector belonging to such an eigenvalue is non-nilpotent of rank 2. A further analysis shows that in symmetric manifolds non-zero critical values of the curvature for bivectors are only possible if the curvature is constant. *H. Freudenthal* (Utrecht)

2382:

Bačurin, G. F. On groups with ascending central series. Mat. Sb. N.S. **45**(87) (1958), 105–112. (Russian)

Theorem 1: If the group G has an ascending central series, and at least one maximal abelian normal divisor A has a finite number of generators, then the group G is nilpotent and has a finite number of generators. Corollary: If the group G has an ascending central series and at least one maximal abelian normal divisor satisfies the maximality (a.c.) condition, for subgroups, then the group G is nilpotent and satisfies the condition of maximality. An example is given of a group which satisfies the d.c. condition for abelian normal divisors, and is not a special group.

Theorem 2: Let G be nilpotent, H a subgroup, $H \neq E$. If H is a complete normal divisor, then H contains a non-trivial complete subgroup in the center of G. Corollary: A quasi-cyclic normal divisor of a nilpotent group is contained in the center.

Theorem 3: If G has ascending central series, and if H is a complete torsionfree normal subgroup, $H \neq E$, then the intersection of H with the center of G is a non-trivial complete group. *J. L. Brenner* (Palo Alto, Calif.)

2383:

Robinson, G. de B. A remark by Philip Hall. Canad. Math. Bull. **1** (1958), 21–23.

Concerning the relationship between representations of symmetric groups and of full linear groups, the author compares the formula for the degree of a symmetric group representation in terms of hook lengths [Frame, Robinson, and Thrall, Canad. J. Math. **6** (1954), 316–324; MR **15**, 931] with a formula for the degree of an irreducible tensor representation. {The latter he seems to attribute to Philip Hall, but it has been frequently used by the reviewer [see, e.g., Proc. Cambridge Philos. Soc. **38** (1942), 394–396; MR **5**, 225].}

He suggests that the formula

$$\{\lambda\} = |\{1+a_s, 1^{b_r}|,$$

where the partition (λ) can be expressed in Frobenius notation as $\binom{a_i}{b_j}$, may have some significance in this connection. *D. E. Littlewood* (Bangor)

2384:

Schützenberger, Marcel Paul. Sur la représentation monomiale des demi-groupes. C. R. Acad. Sci. Paris **246** (1958), 865–867.

Let S be a semigroup. If A [B] denotes the semigroup of mappings of S into S determined by right [left] multiplication by elements of S, then A and B are homomorphic images of S and each element of A commutes with each element of B. These are the basic facts necessary for constructing the author's representation of S by matrices [same C. R. **244** (1957), 1994–1996; MR **19**, 249]. This paper contains a generalization of the previous theory. A and B are here semigroups of mappings of a set Σ^* into itself and such that each element of A commutes with each element of B. Thus A and B are each homomorphic images of a semigroup S. A representation of S by matrices is obtained. {The reviewer cannot see that the 'evident' equation $M(s)M(s') = M(ss')$ of line 17, p. 866 holds without some restrictions upon the $\mathcal{D}$-class Σ involved. Further, the assertion (line 29, p. 866) that the groups $\bar{A}$ and $\bar{B}$ are isomorphic is false (e.g. B may consist of the identity mapping only) and the consequent remarks in the rest of the paper involving the elements of B do not hold without some restrictions on S.} *G. B. Preston* (Shrivenham)

2385:

Tamura, T. Errata: Indecomposable completely simple semigroups except groups. Osaka Math. J. **9** (1957), 241.

The article is in same J. **8** (1956), 35–42 [MR **18**, 282].

2386:

Gluskin, L. M. Matricial semigroups. Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 439–448. (Russian)

Let $G_n{}^r(F)$ be the multiplicative semigroup of all square matrices with elements in the (non-commutative) field F having order n and rank not exceeding r. A semigroup S is isomorphic to $G_n{}^n(F)$ if and only if S contains a dense ideal isomorphic to $G_n{}^1(F)$. The semigroup $G_n{}^1(F)$ is completely simple. The methods developed by the author lead to another proof of the result given by Halezov [Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **5** (1954), 42–56; MR **17**, 825]. *R. A. Good* (College Park, Md.)

2387:

Wiegandt, Richard. On complete semi-groups. Acta Sci. Math. Szeged **19** (1958), 93–97.

A T-semigroup is defined as a semigroup which has some specified property T. A T-semigroup S is called complete with respect to T if it is a direct factor in every T-semigroup which is a Schreier extension of S (that is, in which S is a normal subsemigroup). The author proves that a regular (=cancellation) semigroup F with identity is complete if and only if its automorphisms are all inner automorphisms, and its center consists of the identity. The proof is a modification of R. Baer's proof for the particular case where F is a group [Bull. Amer. Math. Soc. **52** (1946), 501–506; MR **8**, 14], and uses a theorem of L. Rédei [Acta Sci. Math. Szeged **14** (1952), 252–273; MR **14**, 614]. Corollaries are: (1) Every complete soluble and complete nilpotent group is the identity; (2) every finitely generated complete abelian group is the identity. *R. Artzy* (Haifa)

2388:

Scott, W. R. Half-homomorphisms of groups. Proc. Amer. Math. Soc. **8** (1957), 1141–1144.

A map f from one multiplicative system into another is called a half-homomorphism if, for all x and y, either

$f(xy)=f(x)f(y)$ or $f(xy)=f(y)f(x)$. Theorem: Every half-homomorphism of one cancellation semigroup into another is either a homomorphism or an antihomomorphism. Examples are given of half-automorphisms of a semigroup without cancellation and of a loop that are neither automorphisms nor antiautomosphisms.

R. C. Lyndon (Ann Arbor, Mich.)

2389:

Büchi, J. Richard; and Wright, Jesse B. Invariants of the anti-automorphisms of a group. Proc. Amer. Math. Soc. **8** (1957), 1134–1140.

If A is any group, define $G(A)$ to be the group of all automorphisms and antiautomorphisms of A. It was shown by Scott [see preceding review] that if $\alpha(x, y, z)$ is the ternary relation defined by $xy=z$ or $yx=z$, then $G(A)$ is precisely the group of all maps of A onto A that preserve α. Theorem: There is no finite or infinite set of relations $\alpha_\nu(x_1, \cdots, x_{n_\nu})$, defined by equations $w_\nu=1$, where each w_ν is a word in $x_1, \cdots, x_{n_\nu}$, such that, for all groups A, $G(A)$ is precisely the group of all maps of A onto A that preserve all the α_ν. {An alternate proof appears to be contained in the following observations. Permutations f_1, f_2 of the elements of groups A_1, A_2 define a permutation f of their direct product. If f_1 and f_2 preserve a set of equations, so does f. If f_1 is an automorphism and f_2 an antiautomorphism, and A_1, A_2 are not commutative, then f is neither an automorphism nor an antiautomorphism. (Errata: p. 1136, last line, read 'invariance'; p. 1138, sixth line from bottom, read 'four'.)}

R. C. Lyndon (Ann Arbor, Mich.)

2390:

Belousov, V. D. Les quasi-groupes transitifs et distributifs. Ukrain. Mat. Ž. **10** (1958), no. 1, 13–22. (Russian. French summary)

Let M be a quasigroup which is distributive in the sense that $a(bc)=(ab)(ac)$ and $(bc)a=(ba)(ca)$ for all $a, b, c \in M$, and transitive, i.e., there exists a transitive set of permutations φ of the elements of M such that $(\varphi a)b=a(\varphi^* b)$ for a suitable permutation φ^* of M (depending on φ) and for all $a, b \in M$. The main result states that one can introduce a new operation $+$ between the elements of M under which they form an abelian group $M(+)$, and there exist automorphisms ξ and η of $M(+)$ such that $\xi+\eta$ is the identity and $ab=\xi a+\eta b$. The author shows that the quasigroups considered can also be characterized as those satisfying $aa=a$ and $(ab)(cd)=(ac)(bd)$ for all $a, b, c, d \in M$. [Reference should be made to Toyoda, Tôhoku Math. J. **46** (1940), 239–251; MR **2**, 6; where, for quasigroups with the last properties, the main result of this paper is proved.]

L. Fuchs (Budapest)

2391:

Zappa, Guido. Uno sguardo d'assieme alle attuali conoscenze sulla struttura del reticolo dei sottogruppi di un gruppo. Matematiche, Catania **11** (1956), 101–104 (1957).

2392:

Ehresmann, Charles. Gattungen von lokalen Strukturen. Jber. Deutsch. Math. Verein. **60** (1957), Abt. 1, 49–77.

This lecture brings an extended exposition of ideas that have been presented briefly in earlier notes [C. R. Acad. Sci. Paris **234** (1952), 587–589; Ann. Mat. Pura Appl. (4) **36** (1954), 133–142; Colloque de topologie et géometrie differential, Strasbourg, 1952, no. 10, 11, Bibliothèque Nat. et Univ. Strasbourg, 1953; MR **13**, 780; **16**, 504; **15**, 731, 828]. The notions of (mathematical) structure and species of structure are treated in a rigorous and very general way, employing the language of categories and functors. $\mathscr{S}_0$ is a species of structures over C, if $\mathscr{S}_0$ is a class of objects, and C is a groupoid (category with all maps invertible) of operators on $\mathscr{S}_0$. As an example let C be the class of all bijective maps of sets in some class M, and let $\mathscr{S}_0$ be the class of all topologies on sets of M, with an element of C operating on an element of $\mathscr{S}_0$ by transferring the topology to the image set. The further theory involves notions like subspecies, underlying species, extension of species, local categories and species of local structures, pseudo-groups, atlas etc. It is shown how the concepts of local product, foliation, fiber space fit into the theory.

H. Samelson (Ann Arbor, Mich.)

2393:

Ohkuma, Tadashi. Duality in mathematical structure. Proc. Japan Acad. **34** (1958), 6–10.

The structures considered are essentially lattice-ordered categories with zero in the sense of MacLane [Bull. Amer. Math. Soc. **56** (1950), 485–516; MR **14**, 133; p. 505] satisfying three further conditions: (1) the objects are sets, and all images and counterimages of objects are objects; (2) every family of objects has a direct product; (3) the set of mappings from X to Y is an object $\mathrm{Hom}(X, Y) \subset Y^X$. Several theorems are announced, illustrated by Theorem 5: $\mathrm{Hom}(X, \mathrm{Hom}(Z, Y))$ is isomorphic to $\mathrm{Hom}(Z, \mathrm{Hom}(X, Y))$. Fixing a "ground space" L and defining X^* as $\mathrm{Hom}(X, L)$, the author obtains an isomorphism of X^* into X^{***}. The author points out that (3) is invalid in groups and (2) in Banach spaces; in two footnotes he suggests modifications to cover the Banach spaces.

J. Isbell (Seattle, Wash.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 2477, 2633, 2634, 2691, 2695, 2757, 2766.

2394:

Auslander, Louis. A fixed point theorem for nilpotent Lie groups. Proc. Amer. Math. Soc. **9** (1958), 822–823.

Let N be a connected, simply connected nilpotent Lie group and G the group of continuous automorphisms of N. Let A be the semi-direct product of G and N; A acts on N in a natural manner. It is shown that for any compact abelian subgroup B of A, there exists an element n_0 in N such that $bn_0=n_0$ for all b in B.

P. A. Smith (New York, N.Y.)

2395:

Mostert, Paul S. Errata, "On a compact Lie group acting on a manifold". Ann. of Math. (2) **66** (1957), 589.

In this errata the author corrects the error noted in the review of the original paper [Ann. of Math. (2) **65** (1957), 447–455; MR **19**, 44].

A. Shields (Ann Arbor, Mich.)

2396:

Harish-Chandra. Fourier transforms on a semisimple Lie algebra. II. Amer. J. Math. **79** (1957), 653–686.

The purpose of this paper is to complete the proofs of two results (Theorems 2 and 3) which have already been mentioned by the author in a previous paper [same J. **79** (1957), 193–257; MR **19**, 293]. The first of these problems reduces to showing that certain constants are

equal. For this purpose the singular elements have to be taken into account. However it is shown that one needs only certain among them, the so-called semi-regular elements (section 2). Next it is shown that a certain system of differential equations has only the constant solution on a suitable open set. The result (Theorem 2) then follows from the fact that the union of the regular and semi-regular elements is connected. The main results of this paper have already been announced in a short note [Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 538–540; MR 18, 218]. *F. I. Mautner* (Paris)

2397:

Maruyama, Shigeya. Remarks on Haar measure. Kōdai Math. Sem. Rep. 10 (1958), 54–57.

The author proves the following uniqueness theorem for Haar measure. Let G be a locally compact group, K, Z closed subgroups such that $K\cdot Z=G$, $K\cap Z=e$. Let dg be a left invariant Haar measure on G, and let Δ be a continuous homomorphism of G into the multiplicative positive reals such that $dg=\Delta(g)dg^{-1}$. Let $d'g$ be a measure on G with respect to which all continuous functions of compact support are integrable. If $d'kg=d'g$ ($k\in K$) and $d'gz=\Delta(z)d'g$ ($z\in Z$), then $dg=cd'g$ for some constant c. *A. Shields* (Ann Arbor, Mich.)

2398:

Hulanicki, A. Algebraic structure of compact Abelian groups. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 6 (1958), 71–73.

An abelian group is called reduced if it has no non-trivial divisible subgroups. Using results due to Łoś and Kulikoff, the author proves the following theorem: A reduced abelian group admits a compact topology if and only if it is algebraically a complete direct sum of finite cyclic groups and additive groups of p-adic integers. *K. Iwasawa* (Cambridge, Mass.)

TOPOLOGICAL ALGEBRA

2399:

Horne, J. G., Jr. Multiplications on the line. Proc. Amer. Math. Soc. 9 (1958), 791–795.

All associative, continuous multiplications on $E_1=(-\infty,\infty)$ with 0 and 1 playing their natural roles as zero and identity, respectively, and such that $[0,\infty)$ has the usual multiplication, are determined. Any such semigroup is isomorphic to one of the following. (I) For a fixed $\alpha>0$, define $x\circ y=xy$, the usual multiplication of reals, if $y\in[0,\infty)$; $x\circ y=0$ if x and $y\in(-\infty,0)$; and $x\circ y=x^\alpha y$ if $x\in[0,\infty)$, $y\in(-\infty,0)$. (II) Define $x\circ y=y\circ x=xy$ if $x\in[0,\infty)$, $x\circ y=-(xy)$ otherwise. As a consequence, ordinary multiplication is characterized as a semigroup on E_1 with zero and identity, but no other idempotent or nilpotent elements. *P. S. Mostert* (New Orleans, La.)

2400:

Wallace, A. D. Retractions in semigroups. Pacific J. Math. 7 (1957), 1513–1517.

This paper contains what is probably the best result to date in the general theory of topological semigroups. We state the following corollary: if S is a compact Hausdorff topological semigroup, and K is its minimal ideal, then K is a retract of S, and K may be represented topologically and algebraically as a cartesian product of three special subsets of S. (Here one must use a special multiplication on this cartesian product.) This result improves an earlier result of the author [Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 430–432; MR 18, 14].

{The author has pointed out to the reviewer that there is a slight error in the statement of Theorem 3. The set K_e will not in general be a retract of S if e is not an idempotent in the minimal ideal.} *A. Shields* (Ann Arbor, Mich.)

2401:

Šulka, Robert. Remark on isomorphism of topological factoroids. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 6 (1956), 137–142. (Slovak. Russian summary)

A topological groupoid (t.g.) is a topological space G together with a continuous binary operation in G; a topological factoroid (t.f.) on G is a decomposition of G corresponding to an open continuous homomorphism of G and considered itself as a t.g. [cf. R. Šulka, Mat.-Fyz. Časopis. Slovensk. Akad. Vied 5 (1955), 10–21; MR 16, 997]. A simple result is proved which may be stated as follows: a decomposition of a t.f. on a t.g. $\bar{G}$ is a t.f. if and only if it generates a t.f. on G (with which it is, in such a case, necessarily isomorphic). *M. Katětov* (Prague)

2402:

Anderson, Lee W. Topological lattices and n-cells. Duke Math. J. 25 (1958), 205–208.

The author obtains several results about the central elements of a topological lattice. For example, he shows that if L is a locally compact, connected, separable metric, distributive topological lattice with maximal and minimal elements and if the center of L is finite, and if the center of $b\vee(c\wedge L)$ contains no more elements than the center of $a\vee L$ whenever $a\leqq b\leqq c$, then L is homeomorphic and lattice isomorphic to an n-cell. In particular, if L is a locally compact connected topological lattice embedded in the Euclidean plane, and if the center of L is non-void, then L is topologically isomorphic with the unit square. *A. Shields* (Ann Arbor, Mich.)

FUNCTIONS OF REAL VARIABLES

See also 2414, 2588, 2747.

2403:

Salehov, D. V. On Lebesgue-Orlicz points. Dokl. Akad. Nauk SSSR (N.S.) 116 (1957), 355–358. (Russian)

The author considers Orlicz spaces determined by order functions $M(u)$ (for definitions, see W. Orlicz [Bull. Int. Acad. Polon. Sci. A no. 8/9 (1932), 207–220; Zentralblatt Math. 6, 315]). For $p\geqq1$ and a fixed function f in the interval (0, 1), a point x_0 in (0, 1) is a Lebesgue point of order p if

$$\lim_{h\to0}(1/2h)\int_{x_0-h}^{x_0+h}|f(x)-f(x_0)|^p\,dx=0.$$

More generally, for an order function $M(u)$, x_0 is defined to be an L-O (Lebesgue-Orlicz) point of f if

$$\lim_{h\to0}\|M^{-1}(1/2h)[f(x)-f(x_0)]H_h(x)\|_M=0,$$

where H_h denotes the characteristic function of the segment (x_0-h, x_0+h). The following two theorems are typical of the author's results. 1. Suppose that $M_2(u)\leqq M_1(ku)$, for some k and all large u; then, in order that, for each f, all L-O points of order $M_1(u)$ be L-O points of

order $M_2(u)$, it is sufficient that there exist constants k_1 and k_2 such that $M_1(uv) \geq M_1(k_1u)M_1(v)$, $M_2(uv) \leq M_2(k_2u)M_2(v)$, for all large values of the arguments. 2. There exists a measurable function f for which no point is an L-O point of order $M(u)$, where $M(u)$ denotes any order function satisfying a condition $M^2(u) \leq M(ku)$ for all large u. *G. Piranian* (Ann Arbor, Mich.)

2404:

Schmieden, Curt; und Laugwitz, Detlef. Eine Erweiterung der Infinitesimalrechnung. Math. Z. 69 (1958), 1–39.

The purpose of this paper is to show that it is possible to found an Analysis on a new system of numbers, "Ω-numbers" (instead of on the usual system of real numbers), obtained by defining every infinite sequence of rational numbers to have an Ω-number as its limit and defining the limits of two distinct infinite sequences of rational numbers to be distinct Ω-numbers. Certain difficulties in ordinary analysis, for example in connection with rearrangement and multiplication of infinite series, are avoided in Ω-analysis because of the interchangeability of the order of limiting processes. The four rational operations and the relation "$>$" are defined for Ω-numbers in terms of the corresponding operations on or relation between the respective terms of the sequences of rational numbers determining the Ω-numbers. The system of Ω-numbers is shown to be a commutative ring with a unit and divisors of zero, which is not totally ordered, and which contains, with respect to a "weak $>$" relation, infinitely large numbers as well as infinitely small positive numbers. The concepts of function, derivative, and integral are introduced, and some intermediate-value and mean-value theorems are proved. Examples are given of elementary functions and of functions that have the properties of Dirac's δ-function. Infinite series, including power series, are also discussed.

F. Bagemihl (Notre Dame, Ind.)

2405:

Martinot-Lagarde, André. Sur une dérivée généralisée adaptée à l'expérimentation. C. R. Acad. Sci. Paris **246** (1958), 690–692.

The author considers various "generalized derivatives" of a function $f(x)$ of a real variable x, his proposed definitions being related to known methods of estimating the slope of a graph based on results of experiments. His first definition is, in fact, equivalent to that of the ordinary derivative $f'(x)$: he claims that it is more general, but his argument is defective. He considers also the limit $R(x)$ as $h \to +0$ of the integral mean value of $\{f(t+h)-f(t)\}/h$ for $x-h \leq t \leq x$, and he shows that $R(x)$ exists and equals $f'(x)$ if $f'(x)$ exists, but that $R(x)$ may exist when $f'(x)$ does not. *H. P. Mulholland* (Exeter)

2406:

Bögel, K. Die Struktur der stetigen Funktionen einer Veränderlichen. III. J. Reine Angew. Math. **198** (1957), 73–80.

This part III completes the author's memoir [parts I, II are in same J. **196** (1956), 1–33, 137–144; MR **18**, 793]. Here he discusses additional conditions for the continuous function f in order that f be either everywhere differentiable in the linear interval $\bar{J}$ or nowhere differentiable in $\bar{J}$. As a preparation for the second case, a condition is given under which f is nondifferentiable at a "remainder point" x_r (i.e. at a point $x_r \in R(f, \bar{J})$).

A. Rosenthal (Lafayette, Ind.)

2407:

Marcus, Solomon. Sur les dérivées partielles mixtes. C. R. Acad. Sci. Paris **246** (1958), 522–524.

Mixed partial pth derivatives, which exist throughout a Euclidean domain D and differ formally only in the order of the derivations, may be identified in D if they are continuous in each separate variable. The author's proof is a simple application of quasi-continuity in the sense of Kempisty [Fund. Math. **19** (1932), 184–197].

L. C. Young (Cambridge, England)

2408:

Koschmieder, Lothar. Extrema without differential calculus. I, II. Bull. Coll. Arts Sci. Baghdad 2 (1957), 67–83; 3 (1958), 49–49.

Elementary examples, mainly geometric, with historical commentary.

2409:

Cesari, L.; and Turner, L. H. On a lemma in the direct method of the calculus of variations. Rend. Circ. Mat. Palermo (2) 6 (1957), 109–113.

Some years ago, Cesari [Amer. J. Math. **74** (1952), 265–275; MR **14**, 292] improved very beautifully a smoothing process for surfaces, with the help of a simple integral inequality, of which the published proof is adapted to establish a more general inequality with four exponents. (An unpublished simpler calculus proof results from subsequent conversations with the authors.)

L. C. Young (Cambridge, England)

MEASURE AND INTEGRATION

See also 2397, 2668, 2677a-b, 2686, 2777.

2410:

Mickle, E. J.; and Rado, T. Density theorems for outer measures in n-space. Proc. Amer. Math. Soc. **9** (1958), 433–439.

In the Euclidean n-space, let the closed sphere with center at x and radius $r>0$ be denoted by $C(x, r)$, let L_n be the n-dimensional Lebesgue outer measure, let Γ be an ordinary Carathéodory outer measure (i.e., the following assumption must also be satisfied: (*) If the distance between two sets E_1, E_2 is positive, then $\Gamma(E_1 \cup E_2) = \Gamma(E_1)+\Gamma(E_2)$), and let Ψ be a Carathéodory outer measure without assuming (*). Moreover, let E be an L_n-measurable set and let $\mathfrak{C}E$ be the complement of E. Then (1) $\lim_{r\to 0} L_n[E \cap C(x, r)]/r^n = 0$ for L_n-a.e., $x \in \mathfrak{C}E$, and, more generally, (2) $\lim_{r\to 0} \Gamma[E \cap C(x, r)]/r^n = 0$ or ∞ for L_n-a.e., $x \in \mathfrak{C}E$. The purpose of this paper is to prove the following generalization of (2):

$$(3)\quad \limsup_{r\to 0} \frac{\Psi[E \cap C(x,r)]}{r^n} = 0 \text{ or } \infty \text{ for } L_n\text{-a.e., } x \in \mathfrak{C}E.$$

It is also shown that (2) can be derived from (3). Moreover, by an interesting example it is shown that in (3) the limit superior cannot be replaced by the limit.

A. Rosenthal (Lafayette, Ind.)

2411:

Marcus, Solomon. Sur le problème de la mesurabilité des ensembles projectifs. C. R. Acad. Sci. Paris **247** (1958), 21–22.

Announcement of results relating Hamel bases with projective sets, Baire sets, and Lebesgue measure; e.g.: if there exists a Hamel basis that is a projective set, then

there exists a nonmeasurable projective set. Proofs will appear in Math. Nachr. *L. Gillman* (Princeton, N.J.)

2412:

Cafiero, F. Sul teorema di G. Vitali concernente la quasi continuità di una funzione misurabile. Matematiche, Catania **11** (1956), 144–162 (1957).

Let a topological space S be the carrier of a measure space. A function on S is called quasi-continuous if it is continuous in closed sets whose complements have arbitrarily small measure. The author proves that for a finite real-valued function quasi-continuity is equivalent to measurability if S is a perfectly normal topological space and if its measure μ is defined on the μ-completion of the family of all Borel sets of S and is finite. He is unaware of the reviewer's result that quasi-continuity follows from measurability under considerably weaker assumptions [Portugal. Math. **6** (1947), 33–44, 66; **7** (1948), 91–92; MR **9**, 18; **10**, 361].

H. M. Schaerf (Madison, Wis.)

2413:

Tsurumi, Shigeru. On general ergodic theorems. II. Tôhoku Math. J. (2) **9** (1957), 1–12.

[For part I see same J. (2) **6** (1954), 264–273; MR **17**, 136.] Let (X, β, μ) be a finite measure space and let L_1 and L_∞ denote the space of all integrable functions and the space of all essentially bounded functions respectively with the usual norms imposed on them. Let T be a linear positive operator from L_1 into L_1. Let C denote the set of points of X for which $\sum_{i=0}^{\infty} T^i f=\infty$ for every $f\in L_1$. A set A is said to be T-invariant if $T(f\cdot e_A)=Tf$ in A for every $f\in L_1$, where e_A is the characteristic function of A.

The author proves the following theorem. Let T be such that (i) T is a positive linear operator of L_1 into itself, (ii) $|T|\leqq 1$, (iii) $T1\geqq 0$ a.e., (iv) if $uf=Tf/T1$ for every $f\in L_1$ then $uf=f$ for every $f\in L_\infty$ such that $uf\geqq f$ a.e., (v) $u(f\cdot T^j 1)=uf\cdot uT^j 1$ for every $f\in L_\infty$, $j=0, 1, 2, \cdots$. Then $\sum_{i=0}^{n-1} T^i f/\sum_{j=0}^{n-1} T^i g$ converges a.e. to a function $h(x)$ for every $f\in L_1$ and $g>0$ a.e. and $\int_A h(x)g(x)d\mu=\int_A f(x)d\mu$ for every T-invariant set $A\subseteq C$.

This theorem contains the Hopf ergodic theorem for Markov processes [J. Rational Mech. Anal. **3** (1954), 13–45; MR **15**, 636] as well as the Hurewicz ergodic theorem [Ann. of Math. (2) **45** (1944), 192–206; MR **5**, 148]. The method of proof depends mainly on the methods and results of Hopf and also on their modifications used by N. Dunford and J. T. Schwartz [J. Rational Mech. Anal. **5** (1956), 129–178; MR **17**, 987]. *Y. N. Dowker* (London)

2414:

Bielecki, Adam. Remarque méthodologique sur le second théorème de la moyenne. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **10** (1956), 77–80 (1958), (Polish and Russian summaries)

Die Arbeit enthält elementare Beweise der Formeln

$$\int_a^b f(x)g(x)dx=g(a+0)\int_a^\xi f(x)dx+g(b-0)\int_\xi^b f(x)dx$$

bzw.

$$\int_a^b f(x)g(x)dx=g(a+0)\int_a^\xi f(x)dx \quad (\xi\in(a, b))$$

falls im offenen Intervall (a, b) $f(x)$ beschränkt und integrierbar ist, ferner $g(x)$ beschränkt und monoton, bzw. (zweite Formel) positiv und nichtwachsend oder negativ und nichtabnehmend ist. *J. Aczél* (Debrecen)

2415:

Porcelli, Pasquale. On the existence of the Stieltjes mean σ-integral. Illinois J. Math. **2** (1958), 124–128.

The Stieltjes mean σ-integral $M\int f dg$ on $a\leqq x\leqq b$ is defined as the limit by successive subdivisions of the mean sums $\sum_{i=1}^n (f(x_i)+f(x_{i-1}))(g(x_i)-g(x_{i-1}))/2$ [see H. L. Smith, Trans. Amer. Math. Soc. **27** (1925), 491–515]. Here f is assumed to be bounded and g of bounded variation. At any point x of (a, b), define

$$\omega(f, x^+)=\inf_{d>0}\omega(f; (x, x+d)),$$

where $\omega(f; (x, x+d))$ is the oscillation of f on the open interval $(x, x+d)$, and define $\omega(f, x^-)$ similarly on (a, b). Then the existence theorem states that $M\int f dg$ exists if and only if for every $k>0$ the upper g^* content of the set E_k of the x for which $\omega(f, x^+)\geqq k$ or $\omega(f, x^-)\geqq k$, be zero. Here g^* is the total variation function of g and the upper g^* content is the inf of $\sum_1^n (g(b_i)-g(a_i))$ for non-overlapping $[a_i, b_i]$, with $\omega(f, a_i^-)<k$ and $\omega(f, b_i^+)<k$, covering E_k. As corollaries we have that (a) $M\int f dg$ exists if and only if $M\int f dg^*$ exists; (b) if $M\int f dg$ exists then $M\int |f| dg$ exists; and (c) if $M\int f dg$ exists and f and g have no common discontinuities then $\int f dg$ exists in the ordinary Riemann-Stieltjes sense.

T. H. Hildebrandt (Ann Arbor, Mich.)

2416:

Karták, Karel. Ein Satz über die Substitution in Denjoy-Integralen. Časopis Pěst. Mat. **81** (1956), 410–419. (Czech. Russian and German summaries)

Es handelt sich um die Gültigkeit der bekannten Integralformel

$$(*) \qquad \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx=\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

unter verschiedenen Voraussetzungen über das Integral und die Funktionen f und φ. Ist F eine im Intervall $\langle a, b\rangle$ stetige Funktion, so heißt F eine verallgemeinerte absolut stetige Funktion in $\langle a, b\rangle$ im engeren Sinne oder kurz ein "ACG-Funktion", wenn es zu jeder abgeschlossenen Menge $E\subset\langle a, b\rangle$ eine Teilmenge $\Pi\subset E$ gibt, so daß F auf Π absolut stetig ist und wenn die Folge der Schwankungen der Funktion F in aufeinanderfolgenden Intervallen auf der Menge Π konvergiert. Ist dann f eine beschränkte Funktion in $\langle a, b\rangle$ und existiert eine stetige Funktion F mit $F'=f(x)$ abgesehen von höchstens abzählbar vielen Punkten $x\in\langle a, b\rangle$, so gilt (*), wenn φ "ACG_*-Funktion" in $\langle\alpha, \beta\rangle$ ist mit $a\leqq\varphi(t)\leqq b$ für alle $t\in\langle\alpha, \beta\rangle$. Im Falle eines Denjoy-Integrals gilt: ist f Denjoy-integrabel im engeren Sinne in $\langle a, b\rangle$, φ totalstetig und nicht fallend in $\langle\alpha, \beta\rangle$, ferner $a\leqq\varphi(t)\leqq b$ für alle $t\in\langle\alpha, \beta\rangle$, so gilt (*). Ein analoges Resultat für Funktionen f, welche im weiteren Sinne Denjoy-integrabel sind, wurde von G. P. Tolstov bewiesen [Trudy Mat. Inst. Steklov. **35** (1950); MR **13**, 448]. Da nach einen Satz von Hake-Looman-Alexandrov zwischen dem Perronschen und Denjoy-schen Integralbegriff Äquivalenz besteht, läßt sich unter Verwendung eines Ergebnisses von J. Mařík [Časopis Pěst. Mat. **77** (1952), 1–51, 125–145, 267–301, p. 292; MR **15**, 691] noch der folgende Satz beweisen: die Substitutionsgleichung (*) gilt immer dann (jetzt im Falle eines Perronschen Integrals), wenn entweder $\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$ oder $\int_a^b f(x)dx$ existiert, sobald nur f in $\langle a, b\rangle$ definiert, φ in $\langle\alpha, \beta\rangle$ stetig ist mit $a\leqq\varphi(t)\leqq b$ für alle $t\in\langle\alpha, \beta\rangle$ und sobald nur alle Intervalle $\langle\gamma, \delta\rangle\subset\langle\alpha, \beta\rangle$ in endlich viele Intervalle teilbar sind, so daß auf jedem dieser Teilintervalle φ monoton und totalstetig ausfällt.

M. Pinl (Köln)

2417:

Haupt, Otto; und Pauc, Christian Y. Über Erweiterungen von Inhalten durch Adjunktion von Nullsomen. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1957**, 273–290.

g is a Boolean σ-ring; q is a Boolean ring contained in g; and j is a content or nonnegative additive function on q, denoted by $j|q$. A null-extension of $j|q$ to $j^0|q^0$ via the ideal n_0 in g is defined via the conditions $j^0(Q^0)=j(Q+N)=j(Q)$, where $Q \in q$ and $N \in n_0$, and $Q+N=Q \vee N-Q \wedge N$. If $n(j)$ is the null-class in $j|q$, i.e., consists of Q of q such that $j(Q)=0$, then a necessary and sufficient condition that $j|q$ be null-extensible to $j^0|q^0$ via n_0 is that the intersection of n_0 and q be contained in $n(j)$. The property σ-finite on $j|q$ then extends to $j^0|q^0$. If $j|q$ is a measure, i.e., j is σ-additive and q is a Boolean σ-ring, then $j^0|q^0$ is also a measure. The completeness property of measure (M is in q if and only if, for every $\varepsilon>0$, there exist M', M'' such that $M'\leqq M\leqq M''$ and $j(M''-M')\leqq\varepsilon$) transfers from $j|q$ to $j^0|q^0$. If $j'|q'$ is the minimal extension of the content $j|q$ to a measure and if either $j'^0|q'^0$ or $j^{0\prime}|q^{0\prime}$ exists, then the other exists also, they are equal and $n(j'^0)=n(j^{0\prime})=n(j)+n_0$. If $m|z$ is a measure and k is a σ-ideal in g, and if for any Z of z, $m_r(Z)=\inf m(Z-Z\wedge K)$ for all K in $k\frown z$, then $m_r(z)$ is a measure such that $n(m_r)=n(m)=k\frown z$. Relative to integration, we have: if $m|z$ is a σ-finite complete measure such that $k\frown zCn(m)$, $m^0|z^0$ the null-extension of $m|z$ relative to k, D an element in z and f on D a real m^0-integrable [m^0-summable] function, then there exists an m-integrable [m-summable function] g on D which differs from f only on a set K of k such that $\int_D f dm^0=\int_D g dm$.

T. H. Hildebrandt (Ann Arbor, Mich.)

2418:

Bertolini, Fernando. La teoria algebrica della misura e della integrazione, e suo rapporto con la teoria classica. Ann. Scuola Norm. Sup. Pisa (3) **11** (1957), 225–247.

Caratheodory in his book "Mass und Integral und ihre Algebraisierung" [Birkhäuser, Basel und Stuttgart, 1956; MR **18**, 117] proves that there exist Boolean σ-rings which cannot be mapped isomorphically on a system of sets, but points out that M. H. Stone [Trans. Amer. Math. Soc. **41** (1937), 375–481] has shown that such a mapping is possible if the map of $\bigcup_n A_n$, for an infinite set of somas A_n, is permitted to be a set properly containing $\sum_n A_n'$. where the set A_n' is the map of the soma A_n.This paper carries through the resulting modifications in the set-theoretic aspect of measure and integration. Given a basic set X. Then a class $\mathscr{A}$ of subsets of X is said to be fundamental if it is (1) closed under finite union, intersection and difference and (2) satisfies the condition that for A_n of $\mathscr{A}$, there exists a unique minimal set A in $\mathscr{A}$ denoted by $\mathbf{II}_n A_n$ containing $\sum_n A_n$. Then the maximal set in $\mathscr{A}$ contained in $\prod_n A_n$ exists also and is denoted by $\mathbf{\Pi}_n A_n$. A function μ is a measure on $\mathscr{A}$ if $\mu(\emptyset)=0$ and if, when B and A_n belong to $\mathscr{A}$, with $B\subseteq\sum_n A_n$, then $\mu(B)\leqq\sum_{n=1}^{\infty}\mu(A_n)$. The set of measurable subsets is closed under difference as well as $\mathbf{II}_n$ and $\mathbf{\Pi}_n$, and is completely additive in the sense: if A_n are in $\mathscr{A}$ and disjoint and measurable then $\mu(\mathbf{II}_n A_n)=\sum_n \mu(A_n)$.

The notion of point function is replaced by that of fundamental function relative to $\mathscr{A}$, f being fundamental if it is defined on a subset A of $\mathscr{A}$ and if for every real y there exists a set $A_f(y)$ in $\mathscr{A}$, such that $E[f<y]\subseteq A_f(y)\subseteq E[f\leqq y]$. The set of fundamental functions on a set A is closed under the operations $\cup$, $\frown$, addition, subtraction, multiplication, and reciprocals (of non-vanishing functions) and a modification of $\bigcup_n f_n$ and $\bigcap_n f_n$. It is shown that the notion of fundamental function is equivalent to Caratheodory's Ortsfunktion. If $\mathscr{A}_\mu$ is the class of sets measurable relative to a measure function μ, then integration of positive fundamental functions relative to μ on a measurable set A_0 is defined as follows: Let $A_1, \cdots, A_n$ be a finite number of disjoint measurable subsets of A_0 and $y_k\leqq\inf(f(x)|x\in A_k)$. Then $\int f d\mu$ is the supremum of $\sum_{k=1}^n y_k\mu(A_k)$ for all choices of A_k and y_k.

T. H. Hildebrandt (Ann Arbor, Mich.)

2419:

Cesari, Lamberto. A new process of retraction and the definition of fine-cyclic elements. An. Acad. Brasil. Ci. **29** (1957), 1–7.

Fine-cyclic elements are introduced in terms of a new concept of retraction and they lead to a fine-cyclic additivity theorem for the Lebesgue area. The basic definitions and results are given without the proofs, which will all appear in the Riv. Mat. Univ. Parma. {Further extensions are promised by C. Neugebauer and it appears from conversations that the notions have much in common with the work of J. W. T. Youngs. [Cf. Riv. Mat. Univ. Parma **7** (1956), 149–185, 243–253, 283–292, 333–347; MR **19**, 1168].}

L. C. Young (Cambridge, England)

2420:

Krickeberg, Klaus. Distributionen, Funktionen beschränkter Variation und Lebesguescher Inhalt nichtparametrischer Flächen. Ann. Mat. Pura Appl. (4) **44** (1957), 92, 105–133.

The present paper discusses properties and applications of those functions $f(x_1, \cdots, x_n)$ in R^n whose first partial derivatives (in the sense of Schwartz' distribution theory) are measures. The applications concern, e.g., area of non-parametric surfaces, L. C. Young's closed generalized surfaces, Gauss-Green formula. The paper is remarkable in its generality and its consistent use of distribution theory.

If $n=m+l$, $m\geqq1$, $l\geqq1$, $R^n=R^m\times R^l$, let $(x_1, \cdots, x_n)=(x, y)$, $x\in R^m$, $y\in R^l$. If D is any differential operator in R_n involving only the variables $x_1, \cdots, x_m$, then D can be thought of as an operator D_0 in R_m. If $f(x, y)$ is any summable function in R^n and $y\in R^l$, then $f_y(x)$ denotes the function $f(x, y)$ of x only, $x\in R^m$, and $Df=D_0f_y$. Here both D, D_0 are distributions defined as usual by $(Df, \varphi)=(f, D^*\varphi)$, $(D_0f, \varphi)=(f, D_0^*\varphi)$, where D^*, D_0^* are the operators adjoint to D, D_0 and φ denotes any infinitely differentiable function with compact support. If $\mathfrak{B}_n$ denotes the family of all Borel sets $B\subset R^n$, λ_l the Lebesgue measure in R^l, P the operation of projection from R^m into R^l, then Radon measures μ in R^n are considered with the property that $B\in\mathfrak{B}_n$, $\lambda_l(P(B))=0$ implies $\mu(B)=0$. By the use of Halmos' decomposition of measures it is shown that $\mu(B)=(R_l)\int\mu_y(B_y)d\lambda_l(y)$ for every $B\in\mathfrak{B}_n$, where $B_y=[x\in R^m, (x, y)\in B]$, and μ_y a convenient measure in R^m depending on y. By μ_y^+, μ_y^-, μ_y'' are denoted the positive, negative, and total variations of μ_y. I. The following statements are equivalent: (M1) The distribution D_f is a measure μ; (M2) For almost all y the distribution D_0f is a measure μ_y in R_m and, for every $B\in\mathfrak{B}_n$, the total variation $\mu_y''(B_y)$ is a summable function of y in R^l; (M3) As in (M2) with $\mu_y''(B_y)\leqq v(y)$ for some summable function $v(y)$ in R^l. The equivalence of (M2) and (M3) extends a result of the reviewer for $n=2$, $m=l=1$, $D=\partial/\partial x$ [Ann. Sci. Norm. Sup. Pisa (2) **5** (1936), 197–210]. It is assumed now $m=1$, $l=n-1$. For every $i=1, \cdots, n$, let R^1, R^{n-1} be the x_i- and y-spaces, $y=(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$. For every f in R^n, $y\in R^{n-1}$,

let $f_{i(y)}(x_i)=f(x_i, y)$. For every set B let $B_{i(y)} \subset R^1$ be the set indicated above as B_y. The following condition is singled out: (T_i) There is a function f^i in R^n with $f^i=f$ almost everywhere, and f^i is such that, for almost all $y \in R^{n-1}$ and every interval $J \subset R^n$, the function $f^i{}_{i(y)}(x_i)$ is BV in $J_{i(y)}$, and its total variation $V(f^i{}_{i(y)}, J_{i(y)})$ is a summable function of y. A function f is said to be locally of bounded variation in the sense of Tonelli and Cesari (locally BV) provided the n conditions T_i, $i=1, \cdots, n$, hold. II. A function f in R^n, which is locally BV, is locally summable and the n distributions $\partial f/\partial x_i$ are measures μ_i. Conversely, if the n first partial derivatives of a distribution T in R^n are measures, then T is a function f which is locally BV.

In L. C. Young's theory the track of a generalized surface is a vector $\Phi=(\mu_1, \cdots, \mu_n)$ of Radon measures μ_i in R^n with compact supports. If $\varphi=(\varphi_1, \cdots, \varphi_k)$ denotes any vector of functions φ_i indefinitely differentiable in R^n and with compact supports, then Φ is also the operator defined by $(\Phi, \varphi)=(R^n)\int \varphi \cdot d\Phi$. Finally Φ is said to be the track of a closed generalized surface if $(\Phi, \varphi)=0$ for every φ as above with div $\varphi=0$. III. The gradient of a locally BV function with compact support is the track of a closed generalized surface and vice versa. Also, the locally BV functions are those "with generalized gradient" in the sense of de Giorgi. Both results are proved straightforwardly in terms of distribution theory.

If f is locally summable over an open set $G \subset R^n$ let $\alpha(f)$ denote the Lebesgue area of the n-surface S: $z=f(x)$, $x \in G$. Here $\alpha(f)$ is defined by means of the n-dim. elementary areas of quasi linear functions and the use of mean convergence. IV. $\alpha(f)<+\infty$ if and only if $\lambda_n(G)<+\infty$, and f is locally BV. Then $\alpha(f)$ is the total variation in G of the vector valued measure $(\lambda_n, \mu_1, \cdots, \mu_n)$, with $\mu_i=\partial f/\partial x_i$, i.e., $\alpha(f)=(G)\int[(d\lambda_n)^2+(d\mu_1)^2+\cdots+(d\mu_n)^2]^{\frac{1}{2}}$. This theorem extends previous results of C. Goffman [Rend. Circ. Mat. Palermo (2) **2** (1953), 203–235; Amer. J. Math. **76** (1954), 679–688; MR **16**, 457, 458] and the reviewer [loc. cit.]. *L. Cesari* (Baltimore, Md.)

2421:

Fleming, W. H. Functions with generalized gradient and generalized surfaces. Ann. Mat. Pura Appl. (4) **44** (1957), 92, 93–103.

According to the paper reviewed above, the concepts of "gradient of a function $f(x)$, $x=(x_1, \cdots, x_k)$, with a compact support (say a fixed cube $K \subset R^k$) and whose first partial derivatives $\Phi=(\mu_1, \cdots, \mu_k)$ are Radon measures in R^n" and of "track of a closed generalized surface" are proved to be equivalent. In the present paper a new proof of this equivalence is given in terms of L. C. Young's theory, together with the result I. below. Let $\mathfrak{F}$ be the collection of all functions f as above and put $I(f)=V(\Phi)$, the total variation of Φ (thus $I(f)<+\infty$); let $\mathfrak{F}_N$ be the subcollection of all $f \in \mathfrak{F}$ with $I(f) \leq N$. If E is any measurable set $E \subset K$, and f_E the characteristic function of E, then $P(E)=I(f_E)$ is a sort of "measure" of the boundary of E. Thus $P(E)$ is said to be the perimeter of E. Let $\mathfrak{E}$, $\mathfrak{E}_N$ be the collections of all measurable sets $E \subset K$ with $P(E)<+\infty$, $P(E) \leq N$ (Caccioppoli sets). With the usual l_1 topology $\mathfrak{F}_N$ is a compact convex space and hence the convex closure of its extreme points (theorem of Krein-Milman). Also, by a result of G. Choquet, each element of $\mathfrak{F}_N$ has an integral representation in terms of a measure carried by the set of extreme points. I. If $k=2$ or 3, and f is an extreme point of $\mathfrak{F}_N$, then there is a set $E \subset \mathfrak{E}_N$ with $f_E=\pm N^{-1}P(E)f$. For $k=2$, E is a simple Jordan region, $E \subset K$, whose boundary is a rectifiable (simple closed) curve of Jordan length $L=P(E)$. *L. Cesari* (Baltimore, Md.)

2422:

Pauc, C. Y. Functions with generalized gradients in the theory of cell functions. Ann. Mat. Pura Appl. (4) **44** (1957), 92, 135–152.

In a previous paper the author [Ann. Mat. Pura Appl. (4) **40** (1955), 183–192; MR **17**, 1190] had compared Fichera's and de Giorgi's definitions of functions $f(x)$ with "generalized gradients", proving that they are equivalent for continuous functions, and that Fichera's definition is no longer suitable for noncontinuous functions. In the present paper the author rewords both definitions for functions $f(x)$, $x=(x_1, \cdots, x_k) \in R^k$, measurable and summable in R^k with support in a fixed cube $K \subset R^k$, by using partitions P of K into intervals (cells) obtained by using only admissible hyperplanes $x_i=c$, $i=1, \cdots, k$. These hyperplanes are those for which $f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_k)$ is approximately continuous with respect to x_i for almost all $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k)$. These decompositions P had been used previously by C. Goffman [Rend. Circ. Mat. Palermo (2) **2** (1953), 203–235; Amer. J. Math. **76** (1954), 679–688; MR **16**, 457, 458]. It is proved that the modified de Giorgi and Fichera total variations are equal, and thus the two definitions of functions with generalized gradients, so modified, have the same range of validity. A result of C. Goffman [loc. cit.] is used to state that these functions are the same as those which are locally of bounded variation in the sense of Tonelli and Cesari [see review of Krickeberg's paper above]. *L. Cesari* (Baltimore, Md.)

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See also 2546.

2423:

Lammel, Ernst. Eine Bemerkung zum Satz von Vitali über Konvergenz von Funktionenfolgen. Math. Nachr. **18** (1958), 309–312.

The author gives a new proof for Vitali's theorem on the convergence of locally uniformly bounded sequences of analytic functions; this proof is presented for the unit disk $|z|<1$ (from which it follows for a general region). The proof is based on a particular representation of a regular function in $|z|<1$, which was given by the author previously [Časopis Pěst. Mat. Fys. **66** (1936), 57–62]. *A. Rosenthal* (Lafayette, Ind.)

2424:

Yûjôbô, Zuiman. Supplements to my paper: On pseudo-regular functions. Comment. Math. Univ. St. Paul. **4** (1955), 11–13.

The paper referred to appeared in same Comment. **1** (1953), 67–80; MR **15**, 416.

2425:

Strebel, Kurt. Eine Abschätzung der Länge gewisser Kurven bei quasikonformer Abbildung. Ann. Acad. Sci. Fenn. Ser. A. I. no. **243** (1957), 10 pp.

Let the infinite strip G_z: $-\infty<x<\infty$, $0<y<1$ of the $z=x+iy$ plane be mapped quasi-conformally onto a congruent strip G_w of the w-plane by means of the mapping $w_0(z)=K_0x+iy$ $(K_0 \geq 1)$ with maximal dilation K_0. The author proves that if $w(z)$ denotes an arbitrary quasi-conformal mapping of G_z onto G_w with maximal

dilation K which coincides with $w_0(z)$ on the boundary of G_z, then $K \geq K_0$, and the sign of equality holds only if $w(z) \equiv w_0(z)$. This result is established by first obtaining an estimate for the lengths of the images of vertical segments under a quasi-conformal map with maximal dilation $K \geq 1$. Roughly expressed, this inequality means that on the average these images can not be very long if horizontal segments are mapped on curves of K-fold length. As pointed out by the author, the application of this method of estimation is not confined to strips bounded by straight lines.

W. Seidel (Notre Dame, Ind.)

2426:

Specht, Wilhelm. Die Lage der Nullstellen eines Polynoms. II. Math. Nachr. **16** (1957), 257–263.

[For part I, see same Nachr. **15** (1956), 353–374; MR **19**, 400.] For an arbitrary positive constant C and with $B = \max(|a_k|C^{-k})$, $1 \leq k \leq n$, the polynomial $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ is known to have all its zeros in the open disk $|z| < C(1+B)$. In this paper, Specht sharpens this result by taking $B = b_1 - \sum_{k=1}^{n} d_k(1+b_1)^{-k}$, where $d_k = b_k - b_{k-1}$ and where $b_1, b_2, \cdots, b_n$ are the quantities $|a_k|C^{-k}$ arranged in decreasing order of magnitude. He shows further that with a suitable definition of the b_k, the strip $|\mathrm{Im}\, z| < C(1+B)$ contains all the zeros of the function $f(z) = a_0 P_0(z) + a_1 P_1(z) + \cdots + a_n P_n(z)$ where $\{P_k(z)\}$ form a set of polynomials orthonormal with respect to a positive real weight function on an interval of the real axis.

M. Marden (Milwaukee, Wis.)

2427:

Specht, Wilhelm. Die Lage der Nullstellen eines Polynoms. III. Math. Nachr. **16** (1957), 369–389.

In this paper, the author studies the zeros of functions $f(z)$ of the form $f(z) = d_0 p_0(z) + d_1 p_1(z) + \cdots + d_n p_n(z)$ where the polynomials p_k, $k = 0, 1, 2, \cdots$, form a "derivative-sequence" defined by the relations $p_0(z) = 1$, $p_{k+1}'(z) = p_k(z)$, $k = 0, 1, 2, \cdots$. Corresponding to a given sequence $A = (1, a_1, a_2, \cdots)$, let $p_n(z) = p_n(A, z) = (z^n/n!) + [a_1 z^{n-1}/(n-1)!] + \cdots + a_n$ and $P(A, z) = 1 + \sum_1^\infty a_k z^k$; then $F(z, t) = P(A, t)\, e^{zt} = \sum_0^\infty p_k(A, z) t^k$.

The choices $P(A, z) = 1$ and $P(A, z) = e^{-z^2/2}$ lead respectively to $p_n(z) = z^n/n!$ and $p_n(z) = H_n(z)$, the Hermite polynomials, as examples of derivative-sequences. Let the product $C = AB$ of the two sequences A and $B = (1, b_1, b_2, \cdots)$ be defined as $C = (1, c_1, c_2, \cdots)$ with $c_n = \sum_{k=1}^{n} a_k b_{n-k}$. Let $K[f]$ denote the convex hull of all the zeros of f and $U+V$ the set comprised of all points $(u+v)$ for all points $u \in U$ and $v \in V$. Using Grace's theorem for apolar polynomials, the author then shows that $K[p_n(AB, z)] \subseteq K[p_n(A, z)] + K[p_n(B, z)]$, $n = 1, 2, \cdots$.

Among his applications of this theorem is that if for $n \geq 2$ all the zeros of $f(z) = \sum_0^n a_k(z^k/k!)$ be on the disk $|z-s| \leq R$, then all the zeros of the polynomial $F(z) = \sum_0^n a_k H_k(z)$ lie in the strip $|z - t(s-k_n) - (1-t)(s+k_n)| \leq R$, $0 \leq t \leq 1$, where $k_n^2 = 4[n - n^{\frac{1}{2}} - 1]$ and $n \geq 3$.

M. Marden (Milwaukee, Wis.)

2428:

Parodi, Maurice. A propos de la localisation des zéros de la dérivée du polynome caractéristique d'une matrice. C. R. Acad. Sci. Paris **246** (1958), 1131–1133.

The author refers to two theorems [see M. Marden, Geometry of the zeros of a polynomial in a complex variable, Mathematical Surveys, No. 3, American Mathematical Society, New York, 1949; MR **11**, 101; p. 67 and p. 90]: (1) Walsh's theorem that, if an nth degree polynomial $f(z)$ has p zeros in a disk $|z-a| \leq r$ and the remaining $n-p$ zeros in the disk $|z-b| \leq s$, then any zero of its derivative $f'(z)$ not in one of these disks lies in the disk $|z-c| \leq t$, where $c = (1/n)[pb + (n-p)a]$, $t = (1/n)[ps + (n-p)r]$; (2) Marden's theorem that $f'(z)$ has at least $p-1$ zeros in the disk $|z-a| \leq r \csc[\pi/2(n-p+1)]$ if $f(z)$ has p zeros in the disk $|z-a| \leq r$.

The author applies these theorems in conjunction with the result that each characteristic value of the nth order square matrix $A = (a_{ij})$ lies in at least one of the circles $|a_{ii} - z| \leq \sum |a_{ij}|$ (the summation being for $j = 1, 2, \cdots, i-1, i+1, \cdots, n$). He thereby obtains some immediate results on the zeros of the derivative of the characteristic polynomial of A.

M. Marden (Milwaukee, Wis.)

2429:

Roux, Delfina. Media, funzione maggiorante e somme di coefficienti per le serie di potenze di ordine finito. Riv. Mat. Univ. Parma **7** (1956), 187–209.

In the present paper the author makes an extensive study of the relationship between certain classical majorants for functions $f(z) = \sum a_n z^n$ defined by power series having radius of convergence unity. The majorants studied are $M_2(f, r) = \{(1/2\pi)\int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta\}^{\frac{1}{2}} = (\sum_0^\infty |a_n|^2 r^{2n})^{\frac{1}{2}}$; $\mathfrak{M}(f, r) = \sum_0^\infty |a_n| r^n$; $A(f, h) = \sum_0^h |a_n|$; $B(f, h) = (\sum_0^h |a_n|^2)^{\frac{1}{2}}$. Denote, for all $\alpha \geq 0$, by $C(M_2, \alpha)$ the class of functions $f(z)$ for which there exists an $r_0 = r_0(f)$ such that $M_2(f; r) \leq (1-r)^{-\alpha}$ for $r_0 \leq r < 1$; by $C(A, \alpha)$ the class for which there exists an $h_0 = h_0(f)$ such that $A(f; h) \leq h^\alpha$ for $h \geq h_0$. Similar definitions are given for the classes $C(\mathfrak{M}, \alpha)$ and $C(B, \alpha)$. Also if C denotes a class of functions and K any number, we denote by $K \cdot C$ the class of functions whose elements are $K \cdot f(z)$. A constant $K = K_{M,B}(\alpha)$ is said to be an admissible constant for the classes $C(M_2, \alpha)$ and $C(B, \alpha)$ if $f(z) \in C(M_2, \alpha)$ implies that $f(z) \in K_{M,B}(\alpha) C(B, \alpha)$. Similar definitions are introduced for the other classes. This leads to a study of the eight classes of constants $K_{M,B}(\alpha)$, $K_{M,\mathfrak{M}}(\alpha)$, $K_{M,A}(\alpha)$, etc. For each of these eight sets of constants we introduce the notation $K_{M,B}^*(\alpha) = \mathrm{Inf}\, K_{M,B}(\alpha)$, etc. This then leads to a study (for each $\alpha \geq 0$) of the eight constants K^*. The present paper is then devoted to determining the range of values of the constants K^*. The author then finds among other results that

$$2^\alpha/\sqrt{(2\alpha)!} \leq K_{M,B}^*(\alpha) \leq (e/\alpha)^\alpha,$$

$K_{M,\mathfrak{M}}^*(0) = 0$ (the well-known theorem of Hardy [Quart. J. Math. **44** (1913), 147–160]). Many other results of this nature are given.

V. F. Cowling (Lexington, Ky.)

2430:

Wittich, Hans. Zur konformen Abbildung schlichter Gebiete. Math. Nachr. **18** (1958), 226–234.

Application is made of the method of extremal lengths to establish a number of results concerning univalent analytic functions (e.g. Koebe $\frac{1}{4}$-theorem, relation between inner and outer conformal radius, a theorem of Grötsch, etc.).

M. H. Heins (Urbana, Ill.)

2431:

Woods, L. C. Some generalizations of the Schwarz-Christoffel mapping formula. Appl. Sci. Res. B. **7** (1958), 89–101.

Proceeding by analogy with the derivation of the Schwarz-Christoffel formula for the conformal mapping of the interior of a polygonal region in the z-plane onto

the upper half of the w-plane, namely

$$\frac{dz}{dw}=K\prod_{n=1}^{N}(w-\phi_n)^{-\alpha_n/\pi},$$

where the α_n are the measures of the exterior angles of the polygon at the vertices z_n, and the ϕ_n are the coordinates of the points on the real axis in the w-plane onto which these vertices are mapped, the author establishes an integro-differential formula for the conformal mapping of the interior of an arbitrary simply connected region, bounded by a contour Γ having a continuously turning tangent except at a finite number of points where simple discontinuities in the slope of Γ may exist, in the z-plane onto the upper half of the w-plane. The formula given is

$$\frac{dz}{dw}=D\exp\left\{-\frac{1}{\pi}\int_{-\infty}^{\infty}\ln(w-\phi)d\theta(\phi)\right\},$$

where $\theta(\phi)$ is the measure of the slope of Γ at the point z of Γ that is mapped onto the point having coordinate ϕ on the real axis in the w-plane, and where the integral is of Riemann-Stieltjes type.

E. F. Beckenbach (Los Angeles, Calif.)

2432:

Komatu, Yûsaku. Integraldarstellungen für gewisse analytische Funktionen nebst den Anwendungen auf konforme Abbildung. Kōdai Math. Sem. Rep. 9 (1957), 69–86.

The author extends his earlier generalizations of the Schwarz-Christoffel formula so as to include representations of the conformal mapping of a circular ring onto polygonal domains of a Riemann surface possessing branch points.

P. R. Garabedian (Stanford, Calif.)

2433:

Cornea, A. On the behaviour of analytic functions in the neighbourhood of the boundary of a Riemann surface. Nagoya Math. J. **12** (1957), 55–58.

The author gives a new proof of the following theorem of Kuramochi's: If we delete an arbitrary compact set from a Riemann surface of class $O_{HB}-O_G$ $[O_{HD}-O_G]$, then the resulting surface is of class O_{AB} $[O_{AD}]$. The author gives a common proof for the cases B and D which makes use of the fact that in the complement of a compact set on a surface of class $O_{HB}-O_G$ $[O_{HD}-O_G]$ every HB $[HD]$ function is uniquely determined by its boundary values and its "value" at infinity.

H. L. Royden (Zürich)

2434:

Antonyuk, G. K. On the covering of areas for functions regular in an annulus. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **13** (1958), no. 1, 45–65. (Russian. English summary)

Let $\hat{M}$ be the class of functions $w=f(z)$ which are regular in the ring $K(1, R)$: $1<|z|<R$ and satisfy the conditions $|f(z)|\geqq 1$ and $(2\pi i)^{-1}\int_L (f'(z)/f(z))dz\geqq 1$, where L is a closed contour in $K(1, R)$.

For the Riemann surface $\hat{R}$, the image of $K(1, R)$ under $f(z)$, the author defines $\hat{R}^*$, the star of $\hat{R}$, in a manner much too involved for restatement here. (The definition itself covers fully two printed pages.) In this connection, he points out flaws and inconsistencies in the definition of $\hat{R}^*$ given by G. Hažalia [Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18** (1951), 245–256; MR **14**, 549], showing that, with the definition given there, $\hat{R}^*$ need not exist. With the present definition, the author shows that $\hat{R}^*$ exists and then establishes the inequality

$$(*)\quad (1+S/\pi)(1+s/\pi)\geqq R^4,$$

where S is the area of the star and s that of its original. In addition, it is shown that equality is possible in (*) only for $f(z)=\varepsilon z$, $|\varepsilon|=1$.

J. F. Heyda (Cincinnati, Ohio.)

2435:

Tamura, Jirô. On the maximal Riemann surface. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **7** (1957), 19–22.

The following theorem is established and applied to problems concerning non-continuable Riemann surfaces and non-continuable continuations of a given Riemann surface. Let $\mathbf{G}$ denote a family of Riemann surfaces none of which has planar boundary elements such that the following conditions are fulfilled: (a) If $G_n\in\mathbf{G}$, $G_n\subset G_{n+1}$, $n=1, 2, \cdots$, then $\bigcup G_n\in\mathbf{G}$. (b) There exists a bounded real-valued function $\mu(G)$ called the extent of G which is defined for every $G\in\mathbf{G}$ and has the following properties: (i) $G\subset G'$ implies $\mu(G)\leqq\mu(G')$, (ii) if $G\in\mathbf{G}$ is continuable, then there exists a surface $G'\in\mathbf{G}$ such that $G\subset G'$ and $\mu(G)<\mu(G')$. Then there exists a non-continuable surface in the family $\mathbf{G}$.

M. H. Heins (Urbana, Ill.)

2436:

Rosen, David. A note on the behavior of certain automorphic functions and forms near the real axis. Duke Math. J. **25** (1958), 373–380.

The author considers continued fraction type representations of points on the real axis defined as limit points of the group of Hecke [Math. Ann. **112** (1935), 664–699], generated by $T(z)=-1/z$ and $S(z)=z+2\cos\pi/q$, q integral $\geqq 3$. The behavior of automorphic functions connected with the group is then considered in relation to that representation. The methods are extensions of those employed by Bernard Epstein and Joseph Lehner [in unpublished work] and the reviewer [Amer. J. Math. **71** (1949), 403–416; MR **10**, 603].

H. Cohn (Tucson, Ariz.)

2437:

Bertrandias, Françoise. Sur les fonctions analytiques possédant une certaine propriété arithmétique. C. R. Acad. Sci. Paris **247** (1958), 22–24.

The principal results stated in this note are as follows. Let f be analytic and of exponential type in the angle $|\arg z|<\delta$, where $\delta>\pi/2$. The Laplace transform of f is regular outside a set S in the s-plane; let T be the image of S under $z=se^s$. Then if $f^{(n)}(n)$ is an integer when n is a nonnegative integer, and if the transfinite diameter of T is less than 1, it follows that $f^{(n)}(n)=P_1(n)\alpha_1{}^n+\cdots+P_k(n)\alpha_k{}^n$, where P_j are polynomials and α_j are algebraic integers which, with their conjugates, are in T. If S is in the domain with polar equation $\rho<(\pi-\theta)\csc\theta$, then $f(z)=Q_1(z)\exp(z\sigma(\alpha_1))+\cdots+Q_k(z)\exp(z\sigma(\alpha_k))$, where $\sigma(z)$ is the inverse of se^s. In particular, if f is an entire function of exponential type γ and $f^{(n)}(n)$ is an integer, then for $\gamma\leqq.678\cdots$, f is of the above form; if $\gamma<.567\cdots$, f is a polynomial.

R. P. Boas, Jr. (Evanston, Ill.)

2438:

Leont'ev, A. F. Values of an entire function of finite order at given points. Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 387–394. (Russian)

The author establishes the following theorems on the existence of entire functions that assume given values at given points. (1) If $\limsup(\log n)/\log|\lambda_n|\leqq\rho$, there is an entire function of order at most ρ assuming the values a_n at λ_n if and only if $\limsup(\log\log\beta_n)/\log|\lambda_n|\leqq\rho$, where β_n is determined as follows: let $\mu_1{}^{(n)}, \cdots, \mu_{q_n}{}^{(n)}$ be the λ_m lying in $|z-\lambda_n|<|\lambda_n|^{-h}$, and let $\alpha_1{}^{(n)}, \cdots, \alpha_{q_n}{}^{(n)}$ be the

corresponding a_m. Let

$$A_k^{(n)}=\mu_1^{(n)}\cdots\mu_k^{(n)}\sum_{p=1}^{k}\alpha_p^{(n)}/\prod_{j=1}^{k}{}'\,(\mu_j^{(n)}-\mu_p^{(n)}),$$

$k=2, \cdots, q_n$ (where $\prod'$ omits $j=p$); then β_n is the maximum of $|A_k^{(n)}|$. (2) If $\limsup n|\lambda_n|^{-\rho}<\infty$, there is an entire function of growth at most order ρ, finite type, assuming values a_n at λ_n, if and only if $\limsup|\lambda_n|^{-\rho}\log\beta_n<\infty$. The introduction contains a survey of the literature of similar problems.

R. P. Boas, Jr. (Evanston, Ill.)

2439:

Rahman, Q. I. On means of entire functions. Quart. J. Math. Oxford Ser. (2) **7** (1956), 192–195.

Let $f(z)$ be an entire function of order ρ and lower order λ. Let

$$2\pi\mu_\delta(r)=\int_0^{2\pi}|f(re^{i\theta})|^\delta d\theta,$$

$$\pi r^{\kappa+1}m_{\delta,\kappa}(r)=\int_0^r\int_0^{2\pi}|f(xe^{i\theta})|^\delta x^\kappa dx d\theta,$$

and let $L_{\delta,\kappa}$ and $l_{\delta,\kappa}$ be the upper and lower limits of $\{\mu_\delta(r)/m_{\delta,\kappa}(r)\}^{1/\log r}$. The author shows that $L_{\delta,\kappa}=e^\rho$, $l_{\delta,\kappa}=e^\lambda$; only the case δ, $\kappa=2$, 1 was previously known.

R. P. Boas, Jr. (Evanston, Ill.)

2440:

Rahman, Q. I. On means of entire functions. II. Proc. Amer. Math. Soc. **9** (1958), 748–750.

Let $M_\delta(r)=\{\mu_\delta(r)\}^{1/\delta}$, where μ_δ is defined in the preceding review; let $\mathfrak{M}_{\delta,\kappa}(r)=r^{-\kappa-1}\int_0^r x^\kappa M_\delta(x)dx$. The author shows that if $\rho<\infty$ then $\log M_\delta(r)\sim\log M(r)$ $(=\log M_\infty(r))$ for each positive δ; and that if $0<\lambda<\rho<\infty$ then $\log\mathfrak{M}_{\delta,\kappa}(r)\sim\log M_\delta(r)$ for $\delta>0$ and $\kappa\geqq-1$.

R. P. Boas, Jr. (Evanston, Ill.)

2441:

Rahman, Q. I. On the coefficients of an entire series of finite order. Math. Student **25** (1957), 113–121.

Let $l_k r$ denote the kth iterate of $\log r$. The author denotes the upper and lower limits of

$$r^{-\rho}(l_1r)^{-\alpha_1}\cdots(l_kr)^{-\alpha_k}\log M(r)$$

by $T(\rho)$, $\tau(\rho)$; those of

$$e^{-1}\rho^{\alpha_1-1}n(l_1n)^{-\alpha_1}\cdots(l_kn)^{-\alpha_k}|a_n|^{\rho/n}$$

by ϑ, Θ; and those of

$$(\alpha_1-1)^{\alpha_1}\alpha_1^{-\alpha_1}n^{\alpha_1}(l_1n)^{-\alpha_2}\cdots(l_{k-1}n)^{-\alpha_k}\left\{\frac{(\alpha_1-1)}{\log(1/|a_n|)}\right\}^{\alpha_1-1}$$

by u and v. He shows that if $f(z)=\sum a_nz^n$ is of positive finite order ρ, then $\tau(\rho)=\vartheta\geqq\Theta$, with equality if $|a_n/a_{n+1}|$ increases. The case for which all $\alpha_k=0$ is classical. If $1<\alpha_1<\infty$ and f is of order zero, we have $\tau(0)=u\geqq v$, with equality if $|a_n/a_{n+1}|$ increases.

R. P. Boas, Jr. (Evanston, Ill.)

2442:

Srivastav, R. P. On the derivatives of integral functions. Math. Student **25** (1957), 11–15.

Let $f(z)$ be an entire function of order ρ and lower order λ, $f^{(s)}(z)$ be the sth derivative of $f(z)$, and $M(r)$ and $M^{(s)}(r)$ be the maximum moduli of $f(z)$ and $f^{(s)}(z)$ ($s=1, 2, 3, \cdots$), respectively. The author proves six theorems relating the behavior of the functions $M(r)$ and $M^{(s)}(r)$ to the values of ρ and λ. In one case $f(z)$ is assumed to be of regular growth. For example: (1) If, for $r\geqq r_0=r_0(f)>1$, the sequence $M(r)$, $M^{(1)}(r)$, $\cdots$ is non-increasing [non-decreasing], then $\lambda<1$ [$\rho>1$]; (2) if $f(z)$ is of regular growth, then $\lim_{r\to\infty}[\log(r\{M^{(s)}(r)/M(r)\}^{1/s})/\log r]=\rho=\lambda$.

A. G. Azpeitia (Amherst, Mass.)

2443:

Rahman, Qazi Ibadur. A note on the derivatives of integral functions. Math. Student **25** (1957), 21–24.

If $f(z)$ is an entire function of order ρ and lower order λ, and $\mu(r,f)$, $\nu(r,f)$ are its maximum term and rank, respectively, then $\nu(r,f)\leqq r\mu(r,f')/\mu(r,f)\leqq\nu(r,f')$, where f' is the derivative of f. Several consequences and new proofs of known results are presented.

A. G. Azpeitia (Amherst, Mass).

2444:

Clunie, J. The behaviour of integral functions determined from their Taylor series. Quart. J. Math. Oxford Ser. (2) **7** (1956), 175–182.

If $f(z)=\sum_1^\infty a_nz^n$ is an entire function and α a positive number such that $[(n+1)/n]^\alpha|a_n/a_{n+1}|$ is ultimately steadily increasing, then: (1) $M(r,f)<[1+o(1)]\phi(\alpha,r,f)$ where

$$\phi(\alpha,r,f)=\alpha^{-\alpha-1}\Gamma(1+\alpha)e^{\alpha\nu}(r,f)\mu(r,f);$$

(2) $\limsup_{r\to\infty}[\log M(r,f)/\log\mu(r,f)]\leqq1+\alpha^{-1}$; and (3) there exists another entire function $g(z)=\sum_1^\infty b_nz^n$ such that $[(n+1)/n]^\alpha|b_n/b_{n+1}|$ is steadily increasing and (A) $M(r,g)>[1-o(1)]\phi(\alpha,r,g)$ for an infinite sequence of values of r and (B) $\limsup_{r\to\infty}[\log M(r,g)/\log\mu(r,g)]=1+\alpha^{-1}$. By using (2) a previous result of S. M. Shah and S. K. Singh [Proc. Roy. Soc. Edinburgh Sect. A **64** (1954), 80–89; MR **16**, 122] is generalized.

A. G. Azpeitia (Amherst, Mass.)

2445:

Künzi, Hans; et Wittich, Hans. Sur la répartition des points où certaines fonctions méromorphes prennent une valeur a. C. R. Acad. Sci. Paris **245** (1957), 1991–1994.

The theorem of Teichmüller-Wittich-Belinsky in the theory of quasiconformal mappings is used to show that the roots of an equation $f(z)=a$ are situated, asymptotically, on certain logarithmic spirals. Here $f(z)$ is a meromorphic function whose Speiser graph has a finite number of periodic ends.

L. Ahlfors (Cambridge, Mass.)

2446:

Ostrovskiĭ, I. V. A generalization of a theorem of M. G. Kreĭn. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 742–745. (Russian)

Consider meromorphic functions of the form $f(z)=\sum A_k(z-h_k)^{-1}$, where $\sum|\mathfrak{I}(h_k^{-1})|<\infty$ and $\sum|A_k|<\infty$. Kreĭn showed [Izv. Akad. Nauk SSSR. Ser. Mat. **11** (1947), 309–326; MR **9**, 179] that if such a function has no zeros then its Nevanlinna characteristic function $T(r)$ does not exceed $C(r+1)$, where C depends on f. The author shows that when f is allowed to have zeros then $T(r)\leqq N(\theta r, 0)+Cr$ for large r, where $\theta>1$ and C depends on θ (and on f). He also shows that if $\limsup r^{-1}T(r)=\infty$ then there are no values with positive defects for f.

R. P. Boas, Jr. (Evanston, Ill.)

2447:

Ahmad, Mansoor. A note on a theorem of Borel. Math. Student **25** (1957), 5–9.

The paper considers exceptional values of meromorphic functions. Several definitions are used, generalising the definition of Borel and applicable to certain classes of functions of infinite order. A number of cases in which there can be at most one exceptional value (or function of lower order) are derived from Nevanlinna's "second fundamental theorem".

A. J. Macintyre (Cincinnati, Ohio)

2448:

Wittich, Hans. Über die Ableitung einer meromorphen

Funktion mit maximaler Defektsumme. Math. Z. **69** (1958), 237–238.

Let $f(z)$ be a meromorphic function for $|z|<R\leqq\infty$ and satisfy the condition

$$\lim_{r\to\infty}\frac{\log r}{T(r,w)}=0;$$

$$\lim_{r\to R}\frac{-\log(R-r)}{T(r,w)}=0$$

if $R<\infty$. The author proves with the help of the known result

$$N(r,1/w')+\sum_{j=1}^{q}m(r,a_j)+S(r)\leqq T(r,w')$$
$$\leqq N(r,w')+m(r,w)+S_1(r)$$

[see Wittich, Neuere Untersuchungen über eindeutige analytische Funktionen, Springer, Berlin, 1955; MR **17**, 1067] the two relations

$$\Delta(w',0)\liminf_{r\to R}T(r,w')/T(r,w)\geqq\sum_{a\neq\infty}\delta(w,a),$$
$$\{1+\Delta(w',0)-\delta(w',0)\}\sum_{a\neq\infty}\delta(w,a)\leqq$$
$$\Delta(w',0)\{2-\delta(w,\infty)-\vartheta(w,\infty)\},$$

the latter one holding for all functions of finite order and for those functions of infinite order for which there are no exceptional intervals Δ_r. These two relations were proved by S. M. Shah and S. K. Singh [Math. Z. **65** (1956), 171–174; MR **17**, 1193] for functions meromorphic for $|z|<\infty$ and of finite order. *S. M. Shah* (Madison, Wis.)

2449:

Collingwood, E. F. On sets of maximum indetermination of analytic functions. Math. Z. **67** (1957), 377–396.

This paper is an elaboration of earlier notes published by the author [C. R. Acad. Sci. Paris **240** (1955), 1502–1504, 1604–1606; J. London Math. Soc. **30** (1955), 425–428; MR **16**, 916; **17**, 600]. Let $f(z)$ be meromorphic in $|z|<1$ and let $e^{i\theta}$ be a point on $|z|=1$. The cluster set $C(f,e^{i\theta})$ is defined to be the set of all points a for each of which there exists a sequence of points $\{z_n\}$, with $|z_n|<1$, $z_n\to e^{i\theta}$, such that $f(z_n)\to a$. Similarly, if G is a subset of $|z|<1$ whose closure $\bar{G}$ contains $e^{i\theta}$, $C_G(f,e^{i\theta})$ is defined by adding the condition $z_n\in G$ to the conditions on $\{z_n\}$ in the preceding definition. A set G is called a set of maximum indetermination of $f(z)$, or, more briefly, maximal at $e^{i\theta}$, if $C_G(f,e^{i\theta})=C(f,e^{i\theta})$. Let G_0 be a subset of $|z|<1$ such that $\bar{G}_0$ has the single point 1 in common with $|z|=1$, and denote by G_θ the image of G_0 under a rotation about the origin through the angle θ. It is shown that under certain conditions on G_0, the set G_θ is maximal at $e^{i\theta}$ for all points $e^{i\theta}$ of a residual set on $|z|=1$. A number of corollaries of this result are given. The points on $|z|=1$ are classified according to their cluster set properties relative to a function meromorphic in $|z|<1$ and the resulting sets are studied from the point of view of their Baire category. The question of cluster sets on spiral paths in $|z|<1$ which converge to $|z|=1$ is investigated. In particular, it is shown that for certain classes of functions every such spiral is maximal. The paper concludes with some applications to functions defined in more general domains. *W. Seidel* (Notre Dame, Ind.)

2450:

Collingwood, E. F. Addendum: On sets of maximum indetermination of analytic functions. Math. Z. **68** (1958), 498–499.

A correction of the proof of Theorem 7 in the preceding paper. The author notes that the reconstructed proof shows that Theorem 7 and Corollary 3 of the preceding paper are valid for an arbitrary function defined in $|z|<1$. *W. Seidel* (Notre Dame, Ind.)

2451:

Collingwood, E. F. Cluster sets and prime ends. Ann. Acad. Sci. Fenn. Ser. A. I, no. **250/6** (1958), 12 pp.

The author points out that certain theorems on cluster sets of meromorphic functions which he had given in an earlier paper [#2449 above; see especially Theorems 1, 2, 3, 4, 8 and Corollary 3] hold for all complex-valued continuous functions, without regard to analyticity. *G. Piranian* (Ann Arbor, Mich.)

2452:

Schild, Albert. On a class of univalent, star shaped mappings. Proc. Amer. Math. Soc. **9** (1958), 751–757.

Let S denote the class of functions $f(z)=z+\sum_2^\infty a_nz^n$ regular and univalent in $|z|<1$; St the subclass of S consisting of those functions $f(z)$ which are also star-like, i.e., Re $(zf'(z)/f(z))>0$ in $|z|<1$; C the subclass of St for which $f(z)$ is convex, i.e., $\mathrm{Re}\{1+zf''(z)/f'(z)\}>0$ in $|z|<1$. The author investigates the class St^* intermediate between St and C for which Re $(zf'(z)/f(z))\geqq\frac{1}{2}$ in $|z|<1$.

Since $f^2(z)/z$ is in St when $f(z)$ is in St^*, sharp bounds for the modulus of $f(z)$ are easily obtained. Since $2(zf'(z)/f(z))-1$ has a positive real part, sharp bounds for the coefficients a_n follow. In both cases the bounds are the same whether $f(z)$ is in C or in St^*. A sufficient condition that $f(z)$ belong to St^* is that $\sum_2^\infty(2n-1)|a_n|\leqq 1$.

Denote by r_f the least upper bound of the radii r for which Re $(zf'(z)/f(z))\geqq\frac{1}{2}$ on $|z|=r<1$ for a fixed $f(z)$. Let r^* denote the greatest lower bound of r_f for all f in S. The author shows that r^* lies in the range $.301<r^*\leqq\frac{1}{3}$. The method depends essentially upon the Bieberbach Area Theorem and the Inequality of Schwarz.

{The reviewer notes that in the author's statement of the theorem of R. F. Gabriel the inequality $\mathrm{Re}(zf'(z)/f(z))\geqq\frac{1}{2}$ should read Re $(zw_2'(z)/w_2(z))\geqq\frac{1}{2}$, where $f(z)=w_1(z)/w_2(z)$.} *M. S. Robertson* (New Brunswick, N.J.)

2453:

Reade, Maxwell O. Two applications of close-to-convex functions. Michigan. Math. J. **5** (1958), 91–94.

This note contains minor extensions of two results, one due to L. Tchakaloff [C. R. Acad. Sci. Paris **242** (1956), 437–439; MR **17**, 724] and the other due to V. S. Rogozhin [Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. **32** (1955), no. 4, 135–137; MR **17**, 724].

Let $f(z)=\sum_{k=1}^n A_k(z-a_k)^{-1}$, $A_k\geqq 0$, where the poles of $f(z)$ are simple and lie in $|z-z_0|<R$. Then $f(z)$ is univalent in a star-shaped neighborhood of $z=\infty$ that contains the exterior of the circle $|z-z_0|=R\sqrt{2}$. The function $g(\zeta)=f\{z_0+R^2(\zeta-z_0)^{-1}\}$ is univalent in a convex domain containing $|\zeta-z_0|<R/\sqrt{2}$.

The theorem can be extended to functions of the form $f_1(z)=\int_0^{2\pi}(z-e^{it})^{-1}d\alpha(t)$, $g_1(\zeta)=f_1(\zeta^{-1})$, where $\alpha(t)$ is a non-constant monotone function. In this case $g_1(\zeta)$ is close-to-convex for $|\zeta|<\frac{1}{2}\sqrt{2}$.

The second extension is that the function $\phi(z)=\int_0^z e^{-t^2}dt$ is univalent in the domain G bounded by the hyperbolas $Lz^2=\pm\frac{1}{4}\pi$, and $\phi(z)$ maps G onto a close-to-convex domain. The domain G may not be the largest domain of univalence for $\phi(z)$. The reviewer notes that the largest circle $|z|=R$ contained in G has $R=\sqrt{(\pi/2)}$, whereas an earlier result of the author [J. Math. Soc.

Japan 9 (1957), 234–238; MR 19, 642] gives a radius of univalency ρ about $z=0$ which exceeds $\sqrt{(\pi/2)}$. The precise value of ρ has recently been found by E. Kreyszig and J. Todd [Bull. Amer. Math. Soc. 64 (1958), 363–364; MR 20 #5871]. *M. S. Robertson* (New Brunswick, N.J.)

2454:

Abe, Hitosi. On p-valent functions. J. Gakugei Tokushima Univ. 8 (1957), 33–40.

The author proves a number of results in geometric function theory of which the following two are typical.

I. Let $w=f(z)=a_0+a_pz^p+a_{p+1}z^{p+1}+\cdots$ be regular in $|z|<1$. Then we have $|a_p|\leqq 4(|a_0|+l_f)$, where l_f denotes the Lebesgue measure of the set of all positive r for which $|w|=r$ lies entirely in the image D_f of $|z|<1$ by $f(z)$. This was previously proved by the reviewer [J. Analyse Math. 1 (1951), 155–179; MR 13, 545] in the case $p=1$. Other results of the reviewer are extended similarly.

II. Let $f(z)=z^{-p}(1+a_1z+\cdots)$ be meromorphic and p-valent in $|z|<1$. Then the image D_f of $|z|<1$ by $f(z)$ includes either the circle $|w|<\delta$ or $|w|>1/\delta$, where $\delta=\sqrt{5}-2$. The value of δ is best possible. Here the author uses a $\frac{1}{4}$-theorem of Biernacki [Bull. Sci. Math. (2) 70 (1946), 45–51; MR 8, 326].

The author also claims to prove that weakly p-valent functions (for definition see the reviewer's paper or review quoted above) have coefficients a_n, which satisfy $|a_n|=O(n^{2p-1})$. However there is a serious gap in the proof. The inequality on p. 35 1.2 does not follow from the previous analysis, since $d\Theta$ may be negative.

W. K. Hayman (London)

2455:

Itô, Jun-iti. The variation of the sign of the real part of a meromorphic function on the unit circle. Trans. Amer. Math. Soc. 89 (1958), 60–78.

The author investigates functions $f(z)$ meromorphic on $|z|\leqq 1$, with zeros and poles on $|z|=1$, for which the real part of $f(z)$ changes sign $2p$ times on $|z|=1$. He obtains generalizations of earlier results for functions regular in $|z|<1$ due to the reviewer [Amer. J. Math. 58 (1936), 465–472; Ann. of Math. 38 (1937), 770–783; Duke Math. J. 5 (1939), 512–518], the author [Sci. Rep. Tokyo Bunrika Daigaku Sec. A 4 (1944), 107–114; MR 14, 34], A. W. Goodman and M. S. Robertson [Trans. Amer. Math. Soc. 70 (1951), 127–136; MR 12, 691], and also, for functions regular in $0<p\leqq|z|<1$, to the reviewer [Canadian J. Math. 4 (1952), 407–423; MR 14, 460]. The author has previously investigated the case in which $f(z)$ is meromorphic in $|z|\leqq 1$ with assigned zeros and poles in $|z|<1$, but with no poles on $|z|=1$ [Bull. Nagoya Inst. Tech. 3 (1951), 293–305].

The author defines a "negative point", $e^{i\zeta_k}$, to be one for which (i) $\Re f(z)$ changes sign on $|z|=1$, $z=e^{i\zeta_k}$ not a pole, $k=1,\cdots,2p$, and (ii) $\varepsilon(-1)^k\Im f(e^{i\zeta_k})\leqq 0$, $\varepsilon=\pm1$ having previously been assigned in a certain normalization of $f(z)$. All the other points $e^{i\tau_k}$, $k=1,\cdots,2p$, at which $\Re f(z)$ changes sign on $|z|=1$, but distinct from the negative points, are called "positive points". Let $\Re_1(p,m)$ be the class of functions $f(z)$ for which (i) $f(z)$ is meromorphic on $|z|=1$, (ii) $\Re f(z)$ changes sign $2p$ times on $|z|=1$, and (iii) the number of positive points is m. A generalized order of multiplicity of the zeros and poles (too technical to reproduce here) on $|z|=1$ replaces the usual one, in which case the author calls the zeros or poles "closed". Let the number of closed zeros and poles on $|z|=1$ be s_1 and t_1, respectively. Let the number of zeros and poles in $|z|<1$ be s and t, respectively. The author's main theorem is the following. Let q be an integer, $f(z)$ be meromorphic in $|z|<1$ and a member of $\Re_1(p,m)$. Let $f(z)=\sum_{n=0}^{\infty}a_nz^{q+n}$, $a_0\neq 0$, near the origin. Then (1) $m-p=q+s+s_1-t-t_1$,

$$(2)\qquad \min_{-\pi\leqq\theta<\pi}\left|\frac{(e^{i\theta}-z)|a_0|^2}{e^{i(\theta-\alpha)}\bar a_0+ze^{-i\beta}a_0}\right|\leqq\left|\frac{f(z)}{z^qP(z)Q(z)}\right|\leqq \max_{-\pi\leqq\theta<\pi}\left|\frac{e^{i(\theta+\alpha)}a_0+ze^{i\beta}\bar a_0}{e^{i\theta}-z}\right|,$$

$$(3)\qquad |a_n|\leqq\max_{-\pi\leqq\theta<\pi}|\phi_n(\theta)|\ (n=0,1,\cdots),$$

where α, β are known constants depending on the given positions of the zeros and poles in $|z|\leqq 1$ and upon p, m, s, t, s_1, and t_1. $P(z)$, $Q(z)$ are known simple rational functions depending upon the zeros and poles of $f(z)$. $\phi_n(\theta)$ is the coefficient of z^n in the power series expansion about $z=0$ of the rational function $P(z)Q(z)[e^{i(\theta+\alpha)}a_0+ze^{i\beta}\bar a_0](e^{i\theta}-z)^{-1}$. The inequalities 2) and 3) are sharp. As in the previous investigations, the essential idea is to formulate $f(z)$ by means of known rational functions with a function regular in $|z|<1$ with positive real part.

As corollaries and applications the author obtains generalizations of inequalities for the coefficients a_n obtained by Goodman [Proc. Amer. Math. Soc. 2 (1951) 349–357; MR 13, 22], Goodman and Robertson [Trans. Amer. Math. Soc. 70 (1951), 127–136; MR 12, 691], and Robertson [Canadian J. Math. 4 (1952), 407–423; MR 14, 460]. *M. S. Robertson* (New Brunswick, N.J.)

2456:

Šuvalova, È. Z. On a sufficient condition for completeness of a system $\{f^{(n)}(z)\}$. Mat. Sb. N.S. 44(86) (1958), 131–136. (Russian)

The author proves that if $f(z)$ is an entire function of exponential type, such that the density of nonvanishing coefficients in its Taylor series is zero, then the system $\{f^{(n)}(z)\}$ is complete in every circle. This follows from the facts that $\{f^{(n)}(z)\}$ is complete unless $f(z)$ is a linear combination of exponentials with polynomial coefficients, and that a function of the latter form cannot have the density of its nonvanishing Taylor coefficients equal to zero.

R. P. Boas, Jr. (Evanston, Ill.)

2457:

Epstein, Bernard; Greenstein, David S.; and Minker, Jack. An extremal problem with infinitely many interpolation conditions. Ann. Acad. Sci. Fenn. Ser. A. I, no. 250/10 (1958), 9 pp.

Let H denote the Hilbert space consisting of all analytic functions quadratically integrable over a domain D, with inner product $(f,g)=\iint_D f(z)\overline{g(z)}\,dxdy$. The authors are concerned with the following problem: Given a set of pairs of complex numbers $\{z_k, a_k\}$, $z_k\in D$, to minimize (f,f) among all functions in H which satisfy the interpolation conditions (A) $f(z_k)=a_k$. If the set $\{z_k\}$ is infinite, there may fail to exist a function of H satisfying all the conditions (A). However, if at least one such function exists, the existence and uniqueness of the extremal function can easily be proved. In the paper, the following special case is first considered: D is the strip $|\mathrm{Im}\,z|<\sigma$, and $z_k=k\pi$, $k=0,\pm1,\pm2,\cdots$. Making use of the isometry between H and $\hat H$, the set of Fourier transforms of elements of H, it is proved that (A) has a solution if and only if $\sum|a_k|^2<\infty$. Assuming that this condition is fulfilled, the authors construct an explicit expression for the sought extremal function, in terms of its Fourier transform. Finally, the method is modified so as to be applicable to somewhat more general domains.

O. Lehto (Helsinki)

2458:

Katznelson, Yitzhak. Sur le problème "$M(r)$". C. R. Acad. Sci. Paris **246** (1958), 211–213.

Le problème "$M(r)$" consiste à trouver des conditions $\mathscr{R}_H\{M(r), E\}$ qui peuvent être définies de la manière suivante (ici $M(r)=\inf M_n r^{-n}$): une condition $\mathscr{R}_H$ et le fait qu'une fonction $\Phi(z)$, holomorphe et uniforme hors de E ($E\subset R$), satisfait aux inégalités $|\Phi(z)z^n y|\leq M_n$ entrainent $\Phi(z)\equiv 0$.

L'auteur indique des conditions $\mathscr{R}_H$ plus fines que celles connues, ainsi qu'un exemple montrant que, pour des ensembles E assez "clairsemés", on ne peut pas les améliorer beaucoup. *S. Mandelbrojt* (Paris)

2459:

Azpeitia, A. G. Convergence of sequences of complex terms defined by iteration. Proc. Amer. Math. Soc. **9** (1958), 428–432.

The reviewer has proved [Acta Univ Szeged. Sect. Sci. Math. **13** (1949), 136–139; MR **11**, 511] that the sequence (1) $a_n=m(a_{n-1}, a_{n-2}, \cdots, a_{n-p})$ is convergent if $m(x_1, x_2, \cdots, x_p)$ is a continuous real function strictly increasing with respect to all x_k and reflexive; i.e., $m(x, x, \cdots, x)\equiv x$. The author extends this result to complex functions, proving that the sequence (1) with initial terms chosen in a convex region D of the complex plane is convergent if the complex function $m(z_1, z_2, \cdots, z_p)$ is continuous in D and if $m(z_1, z_2, \cdots, z_p)$ lies in the convex hull of the points $z_1, z_2, \cdots, z_p$ and is different from the vertices of this convex hull. The proof of this nice result is achieved by geometric considerations on convex polygonal sets and their limits. {The reviewer has also shown, in the paper mentioned above, that the special case

$$m(x_1, x_2, \cdots, x_p)= (a_1x_1+a_2x_2+\cdots+a_px_p)/(a_1+a_2+\cdots+a_p) \quad (a_k>0;\ k=1, 2, \cdots, p)$$

is equivalent to the Eneström-Kakeya theorem [see, e.g., Tôhoku Math. J. **2** (1912), 140–142] on roots of polynomials with decreasing positive coefficients, asserting that they lie in the unit circle. It would be interesting to apply the author's present result to the roots of polynomials with complex coefficients.}

J. Aczél (Debrecen)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 2987.

2460:

Kaizuka, Tetsu. Note on Landau's theorem. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1956), 280–282.

L'A. osserva che il teorema di Landau sui valori eccezionali può trasportarsi ad una funzione di n variabili complesse $w(z_1, \cdots, z_n)$ in una forma che esprime in sostanza il teorema stesso per le funzioni $w(0, \cdots, 0, z_j, 0, \cdots, 0)$. Se ne deduce un'estensione del teorema di Picard alle trasformazioni intere $w_j=w_j(z_1, \cdots, z_n)$ $(j=1, \cdots, n)$ che può ravvicinarsi ad altra stabilita dall'Autore [gli stessi Rep. **5** (1956), 155–157; MR **19**, 644].

E. Martinelli (Rome)

2461:

Bergman, Stefan. A class of pseudo-conformal and quasi-pseudo-conformal mappings. Math. Ann. **136** (1958), 134–138.

The author first states a rather simple distortion theorem concerning the conformal mapping of a square, and then develops an analogous theorem relating to pseudo-conformal (PC) mappings (i.e., schlicht mappings by pairs of analytic functions of two complex variables). Finally, a similar result is obtained for a class of mappings ("quasi-pseudo-conformal" QPC) satisfying a condition similar to one satisfied by PC mappings.

Bernard Epstein (Philadelphia, Pa.)

2462:

Tsuboi, Teruo. Note on complete elements and kernel functions in several complex variables. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1956), 267–279.

The author rewrites the customary definition of the Bergman kernel function [Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la théorie des fonctions analytiques, Mémor. Sci. Math., no. 106, Gauthier-Villars, Paris, 1947; MR **11**, 344] in a matrix notation, and calculates it in some simple cases in 2 complex variables. *R. C. Gunning* (Princeton, N.J.)

2463:

Rizza, G. B. Su diverse estensioni dell'invariante di E. E. Levi nella teoria delle funzioni di più variabili complesse. Ann. Mat. Pura Appl. (4) **44** (1957), 73–89.

On définit un tenseur

$$T_{\rho_1\cdots\rho_{n-2}\bar{\tau}_1\cdots\bar{\tau}_{n-2}}=(-1)^{\rho+\bar{\tau}+1}\begin{vmatrix} 0 & \varphi_{\rho_{n-1}} & \varphi_{\rho_n} \\ \varphi_{\bar{\tau}_{n-1}} & \varphi_{\rho_{n-1}\bar{\tau}_{n-1}} & \varphi_{\rho_n\bar{\tau}_{n-1}} \\ \varphi_{\bar{\tau}_n} & \varphi_{\rho_n} & \varphi_{\rho_n\bar{\tau}_n} \end{vmatrix}$$

dont chaque composante est construite d'une manière analogue à l'invariant $L(\varphi)$ de E. E. Levi, à partir d'une fonction φ supposée deux fois dérivable; T est hermitien; $T=0$ caractérise les hyperplans $\varphi=0$. A partir de la somme des composantes, on définit une expression qui généralise $L(\varphi)$ et permet d'étendre au cas $n>2$, dans l'étude de la pseudo-convexité, la forme des énoncés de E. E. Levi. *P. Lelong* (Paris)

2464:

Erwe, Friedhelm. Einheitsfunktionenspalten. Monatsh. Math. **61** (1957), 173–194.

On considère des vecteurs $f(z)=\{f_1(z), \cdots, f_n(z)\}$ dont les composantes sont des fonctions $f_k(z)$ holomorphes de z pour $|z|<1$, [cf. Peschl et Erwe, Math. Ann. **126** (1953), 185–220; Arch. Math. **4** (1953), 191–201; MR **15**, 520, 21]. On pose $|f|=\sum^n |f_i|^2$. On désigne par U_n la classe des vecteurs $f(z)$ pour lesquels $|f(z)|\to 1$ quand $|z|\to 1$, par H_n celle des vecteurs continus pour $|z|\leq 1$, holomorphes et ne s'annulant pas pour $|z|<1$; f et g dans H_n sont dits équivalents s'il existe une fonction $a(z)\in H_1$, $a(z)\neq 0$ pour $|z|<1$, telle qu'on ait $g(z)=a(z)f(z)$ pour $|z|\leq 1$. A tout vecteur $f(z)\in H_n$ on peut faire correspondre un vecteur $g(z)\in U_n$, équivalent à $f(z)$; il est unique à une constante multiplicative près. Un système $\{f_2, \cdots, f_n\}$, $\sum |f_k|^2<1$ pour $|z|\leq 1$, de $n-1$ fonctions holomorphes se laisse compléter par une composante $f_1(z)$ holomorphe dans $|z|\leq 1$, de manière à former les composantes d'un vecteur $f(z)\in U_n$, et tout vecteur $f(z)\in U_n$ s'obtient par ce procédé, à une rotation près dans C^n du vecteur $f(z)$. On étudie les $f\in U_n$ qui possèdent des composantes f_k rationnelles. Elles s'obtiennent comme dans le cas $n=1$ par une formule de récurrence.

P. Lelong (Paris)

2465:

Lelong, Pierre. Intégration sur un ensemble analytique complexe. Bull. Soc. Math. France **85** (1957), 239–262.

Es war seit langem ein Bedürfnis in der Funktionentheorie mehrerer Veränderlicher, eine exakte Definition des über eine analytische Menge A erstreckten Integrales $\int_A \varphi$ zu besitzen. Da analytische Mengen i.a. nicht uniformisierbare Punkte aufweisen, sind natürliche Schwierigkeiten gegeben. Der Verf. der vorl. Arbeit definiert $t(\varphi)=\int_A \varphi$ und zeigt, daß $t(\varphi)$ ein geschlossener, positiver Courant vom Doppelgrad (p, p) ist ($p=\dim A$). Das Resultat wird mit Hilfe an sich interessanter Fortsetzungssätze für Courants gewonnen. Ein besonderes Interesse des Verf. ist dem Flächeninhalt analytischer Mengen gewidmet. *H. Grauert* (Princeton, N.J.)

2466:

LeLong, Pierre. Integration of a differential form on an analytic complex subvariety. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 246–248.

In der vorl. Note werden Eigenschaften positiver Courants betrachtet. Sie dient im wesentlichen der Ankündigung der Resultate der soeben referierten Arbeit. *H. Grauert* (Princeton, N.J.)

2467:

Lelong, Pierre. Sur l'aire des ensembles analytiques complexes. Ann. Acad. Sci. Fenn. Ser. A. I. no. **250/21** (1958), 10 pp.

In der vorl. Arbeit werden weitere Ergebnisse über positive Courants und den Flächeninhalt analytischer Mengen angekündigt. Der Verfasser betrachtet insbesondere positive Distributionenformen $\Theta=i\sum \Theta_{p\bar{q}}dz_p\wedge d\bar{z}_q$ im komplexen Zahlenraum C^n, die in einem genauer definierten Sinne ein beschränktes Wachstum haben. Es gelingt ihm durch ein Integral eine plurisubharmonische Funktion V zu konstruieren, so daß $\partial\bar{\partial}V=\Theta$ ist. Für V wird eine Abschätzung des Wachstums angegeben. Als Spezialfall ergibt sich eine Konstruktion von holomorphen Funktionen zu vorgegebenen Nullstellenflächen $A\subset C^n$ (Cousin II). Der Verf. gibt an, daß sich das dadurch gewonnene Resultat etwas von dem älteren Kneserschen Satz unterscheidet. *H. Grauert* (Princeton, N.J.)

2468:

Hitotumatu, Sin. Note on the holomorphy on an analytic subset. J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1958), 605–613.

Der Verf. versucht auf beliebigen abgeschlossenen Teilmengen X von Teilgebieten D des n-dimensionalen komplexen Zahlenraumes C^n den Begriff der holomorphen Funktion zu erklären (D-Holomorphie). Nach H. Cartan ist jeder analytischen Menge $A\subset D$ ein Parameterraum A' zugeordnet, der durch eine stetige, eigentliche, fast überall eineindeutige, surjective Abbildung α auf A bezogen ist. Eine stetige komplexwertige Funktion f auf A' heißt bekanntlich eine schwach holomorphe Funktion auf A, wenn $f\circ\alpha^{-1}$ in den gewöhnlichen Punkten von A holomorph im üblichen Sinne ist. Der Verf. zeigt: (1) Ist $X=A$ eine (lokal) irreduzible analytische Menge, so ist eine stetige Funktion auf A genau dann schwach holomorph, wenn sie D-holomorph ist; (2) Ist $X=A$ eine beliebige analytische Menge, f eine beliebige schwach holomorphe Funktion auf A, so läßt sich f in jedem Punkt $P\in A$ in eine n-dimensionale Umgebung zu einer meromorphen Funktion fortsetzen. *H. Grauert* (Princeton, N.J.)

2469:

***Schwartz, Laurent. Variedades analiticas complejas. [Complex analytic manifolds.]** Departamento de Matemáticas y Estadística, Universidad Nacional de Colombia, Bogotá, D.E., Colombia. Course given in July-October, 1956. i+88 pp. (mimeographed) $1.00.

These notes form an excellent introduction to the modern theory of complex analytic manifolds. Definitions and theorems are carefully stated and are motivated by examples. The author refers to the literature for the proofs of most of the basic results. Among the topics discussed are differential forms, currents, $\bar{z}$-cohomology, the Cousin problems, finiteness and duality theorems, and the Riemann-Roch theorem. *J. J. Kohn* (Waltham, Mass.)

2470:

Asami, Takeo. On the conditions of a Stein variety. Osaka Math. J. **9** (1957), 215–219.

Der Verf. beweist folgenden Satz: Es sei X ein (normaler) komplexer holomorph-konvexer Raum. Dann ist X ein holomorph-vollständiger Raum genau dann, wenn es auf X eine positiv definite Levische Funktion gibt. — Der Begriff der positiv definiten Levischen Funktion wird in der Arbeit definiert. Er stellt eine Verallgemeinerung des Begriffes der streng plurisubharmonischen Funktion dar [vgl. auch: H. Grauert und R. Remmert, Math. Z. **65** (1956), 175–194; MR **18**, 475]. *H. Grauert* (Princeton, N.J.)

2471:

Hitotumatu, Sin. On the definition of an analytic function of several complex variables. Sûgaku **8** (1956/57), 25. (Japanese)

The author remarks that we may define an analytic function of several complex variables in the following manner: $f(z)$, $z=(z_1, \cdots, z_n)$, is analytic in a domain Ω if, at each point $a=(a_1, \cdots, a_n)$, there are a set of constants $(\alpha_1, \cdots, \alpha_n)$ and a function $\varepsilon(z; a)$ defined in Ω such that $\lim_{z\to a} \varepsilon(z; a)/\sum_{j=1}^n |z_j-a_j|=0$ and $f(z)=f(a)+\sum_{j=1}^n \alpha_j(z_j-a_j)+\varepsilon(z; a)$. *M. Ohtsuka* (Hiroshima)

2472:

Atiyah, M. F. On analytic surfaces with double points. Proc. Roy. Soc. London. Ser. A **247** (1958), 237–244.

A nodal variety V is a complex n-variety ($n\geqq 2$) with at most a finite number of isolated singularities (nodes), each of which is locally isomorphic to a neighborhood of the vertex of a non-degenerate quadric cone. A family (V, f, R) of $(n-1)$-dimensional nodal varieties consists of a nodal n-dimensional variety V and a holomorphic map $f: V\to R$ of V onto a Riemann surface R, such that, for each $t\in R$, $V_t=f^{-1}(t)$ is a compact, connected, nodal variety. Each node of any given V_t arises either a) from a node of V, at such a point f must be non-critical; or b) from a critical point (necessarily non-degenerate) of f, the latter are assumed to be finite in number. For $n=3$ a special method to "resolve" the nodes of a family (V, f, R) of nodal surfaces yields successively the following. Theorem 2: If (V, f, R) is a family of nodal surfaces, then there exist a family $(\tilde{V}, \tilde{f}, \tilde{R})$ of surfaces without singularities, and two maps, $\alpha: \tilde{V}\to V$, and $\beta: \tilde{R}\to R$ such that: (i) $\beta\tilde{f}=f\alpha$; (ii) β is a 2-sheeted, "branched covering" of R; (iii) for each $s\in\tilde{R}$, the surface $\tilde{V}_s={}^{-1}(s)\tilde{f}$ is the standard non-singular model of $\alpha(\tilde{V}_s)=f^{-1}(\beta(s))$. Theorem 3: Under the assumptions of Theorem 2, for any $a, b\in R$, the standard non-singular models of V_a and V_b are differentiably homeomorphic. Theorem 4: If W is any 3-dimensional nodal, projective algebraic variety, then the standard non-

singular model of any hyperplane section of W that is a nodal surface is differentiably homeomorphic to any non-singular hyperplane section. As a special case of the latter, the featured result is Theorem 1: For each $k=1, 2, \cdots$, the standard non-singular models of the nodal surfaces of degree k in the complex projective 3-space are differentiably homeomorphic to the non-singular surfaces of degree k. This last result is false for algebraic varieties of other dimensions. *E. Calabi* (Princeton, N.J.)

2473:

Nickerson, H. K. On the complex form of the Poincaré lemma. Proc. Amer. Math. Soc. **9** (1958), 183–188.

Sur une variété analytique complexe M, soit $\bar\partial$ la partie de type (0, 1) de l'opérateur de différentiation extérieure. Théorème: Si Φ est une forme différentielle de type (p, q) $(q>0)$ sur un ouvert U de M telle que $\bar\partial\Phi=0$ sur U, alors, au voisinage de tout point de U, il existe une forme Ψ de type $(p, q-1)$ telle que $\Phi=\bar\partial\Psi$. Ce résultat, analogue au lemme de Poincaré, est dû à A. Grothendieck [voir la démonstration dans J-P. Serre, Séminaires H. Cartan, 1953–1954, Math. Dept., Mass. Inst. Tech, 1955, exposé XVIII; MR **18**, 69]; une autre démonstration est due à H. Cartan [voir P. Dolbeault, Ann. of Math. (2) **64** (1956), 83–130; MR **18**, 670]. L'auteur donne une troisième démonstration, en construisant un opérateur local κ, de type $(0, -1)$, sur les formes différentielles, tel que $\bar\partial\kappa+\kappa\bar\partial$ soit l'identité; alors, dans les conditions du théorème: $\Psi=\kappa\Phi$. Si $p=0$, alors: $\kappa\bar\partial\Phi=\Phi-J\Phi$ où $J\Phi$ est une p-forme holomorphe. La technique est une généralisation de celle qu'utilisent Newlander et Nirenberg [Ann. of Math. (2) **65** (1957), 391–404; MR **19**, 577] dans le cas $p=0$, $q=1$.

P. Dolbeault (Paris)

2474:

Morimoto, Akihiko. Sur le groupe d'automorphismes d'un espace fibré principal analytique complexe. Nagoya Math. J. **13** (1958), 157–168.

Let P be a principal complex analytic fibre bundle over a compact complex manifold M with a complex Lie group G as structure group. An automorphism of P is a complex analytic homeomorphism of the bundle space which commutes with the operation of G. The author proves that the group of all automorphisms of P — endowed with the compact open topology — is a complex Lie group. As a corollary it follows that, if P has a complex analytic connection, every complex analytic homeomorphism of M can be lifted to an automorphism of P. The proof is based on the work of Bochner and Montgomery [Ann. of Math. (2) **46** (1945), 685–694; **47** (1946), 639–653; MR **7**, 241; **8**, 253].

M. F. Atiyah (Cambridge, England)

2475:

Beklemishev, D. V. On strongly minimal surfaces of the Riemannian space. Dokl. Akad. Nauk SSSR (N.S.), **114** (1957), 256–258. (Russian)

A class of minimal surfaces is introduced, which are in a close relation to complex-analytical surfaces of corresponding number of dimensions, as they are the two-dimensional minimal surfaces to the complex-analytical surfaces of two dimensions.

The "strongly minimal surfaces of the Riemannian space" are defined by means of certain recurrent relations which must be satisfied by the metric tensor and the so-called "fundamental geometric object of the second kind", which was defined by G. F. Laptev [Trudy Moskov. Mat. Obšč. **2** (1953), 275–382; MR **15**, 254]. The conclusions are derived in connexion with Laptev's results and represent in a certain sense an example of their applicability.

There are quoted four theorems with only indications of proofs. *T. P. Anđelić* (Belgrade)

2476:

Yano, Kentaro. Sur un théorème de M. Matsushima. Nagoya Math. J. **12** (1957), 147–150.

The following theorem of Y. Matsushima [Nagoya Math. J. **11** (1957), 145–150; MR **20** #995] is proved: Let V be a compact complex manifold of Kähler-Einstein. Any analytic contravariant vector u on V can be written, in one and only one way, in the form $u=v+Fw$, where v and w are Killing vectors, and F is the tensor defining the complex structure of V. *E. Vesentini* (Princeton, N.J.)

2477:

Hano, Jun-ichi. On Kaehlerian homogeneous spaces of unimodular Lie groups. Amer. J. Math. **79** (1957), 885–900.

Homogeneous Kähler manifolds, i.e., Kähler manifolds on which a Lie group G acts transitively as a group of complex analytic homeomorphisms preserving the Kähler metric, have been studied by A. Borel [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 1147–1151; MR **17**, 1108] and J. L. Koszul [Canad. J. Math. **7** (1955), 562–576; MR **17**, 1109] under the assumption that G is semi-simple, and by Y. Matsushima [Nagoya Math. J. **11** (1957), 53–60; MR **19**, 315] under the assumption that G is reductive. In the present paper the author makes the weaker assumption that G is unimodular (and connected), then denoting by B the isotropy subgroup, i.e., the subgroup leaving a point x_0 of the manifold fixed, he is able to prove: (1) if B is closed, connected, and semi-simple, the Kähler manifold is locally flat, i.e., its curvature is identically zero; and (2) if B=identity, i.e., if G has a left invariant Kähler structure, then it is meta-abelian; in particular if it is nilpotent, then it is abelian. (3) If the Ricci curvature is assumed non-degenerate and the group acts effectively, then the groups must be semi-simple.

From this last result, applying a theorem of Borel and Koszul [loc. cit.] it follows at once that if a bounded domain in space C^n of complex n-tuples $(z^1, \cdots, z^n)$ admits a transitive, connected unimodular Lie group of homeomorphisms, i.e., if it is a homogeneous space of a unimodular Lie group, then it is symmetric in the sense of E. Cartan [Abh. Math. Sem. Univ. Hamburg **11** (1935), 116–161], thus giving a partial answer to the conjecture of Cartan that a bounded homogeneous domain (of any Lie group) in C^n is necessarily symmetric.

W. M. Boothby (Evanston, Ill.)

SPECIAL FUNCTIONS

See also 2561, 2594, 2853, 2854, 2855.

2478:

Grün, Otto. Beziehungen zwischen Bernoullischen Zahlen. Math. Ann. **135** (1958), 417–419.

A complicated proof that, with the usual umbral notation, the Bernoulli numbers satisfy $(B+B)^n = nB_{n-1}-(n-1)B_n$. The author suggests that interesting arithmetic results follow by combining this recurrence with Von Staudt's theorem. {There are misprints in formulae 1 and 8.} *L. Moser* (Edmonton, Alta.)

2479:
Sharma, A. q-Bernoulli and Euler numbers of higher order. Duke Math. J. **25** (1958), 343–353.

It is not easy to extend the q-Bernoulli and Euler numbers of Carlitz [same J. **15** (1948), 987–1000; MR **10**, 283] to higher order because a simple generating function for these numbers is not known. The present author proceeds as follows. Let h, k be positive integers $\geqq 1$ and let the numbers $\beta_m^{(h,k)}$ be defined for $m \geqq 0$ by

$$\beta_m^{(h,k)} = \beta_m^{(h-1,k)} + (q-1)\beta_{m+1}^{(h-1,k)},$$

$$\beta_m^{(0,k+1)} = \frac{m-k}{[m-k]}\beta_m^{(0,k)}, \quad [x] = \frac{q^x-1}{q-1}.$$

Then $\beta_m^{(0,1)}$, $\beta_m^{(1,1)}$ are equal respectively to the numbers η_m, β_m introduced by Carlitz. Furthermore, for $h=k$ the numbers $\beta_m^{(h,k)}$ reduce to the Bernoulli numbers $B_m^{(k)}$ as $q \to 1$. The author also defines the corresponding numbers $\varepsilon_m^{(h,k)}$ which for $h=k$ and $q=1$ reduce after suitable modification to $E_m^{(k)}$, the Euler numbers of higher order. The first half of this paper is devoted to a detailed study of certain polynomials $\beta_m^{(h,k)}(x)$ and $\varepsilon_m^{(h,k)}(x)$ in q^x. These polynomials lead in the case $h=k$ to the polynomials $\beta_m(x)$ and $\varepsilon_m(x)$ of Carlitz. Generating functions and interrelating formulas are derived. The second half is concerned with an analogous extension of the rational functions $H_m(x, q)$ first investigated by Carlitz in another paper [Trans. Amer. Math. Soc. **76** (1954), 332–350; MR **15**, 686]. *A. L. Whiteman* (Los Angeles, Calif.)

2480:
Carlitz, L. Expansions of q-Bernoulli numbers. Duke Math. J. **25** (1958), 355–364.

The author defines q-Bernoulli polynomials of unrestricted order z by means of the formula

$$\beta_n^{(z)}(x, q) = (q-1)^{-n}\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\left(\frac{r+1}{[r+1]}\right)^z q^{rx},$$

where $[r] = (q^r-1)/(q-1)$. Employing known properties of Bernoulli polynomials $B_n^{(k)}(x)$ of higher order he establishes for $n \geqq 1$ that

$$\beta_n(q) = \beta_n^{(1)}(0, q) = \sum_{k=n}^{\infty}(q-1)^k\sum_{s=n}^{k}\frac{\beta_s}{s!}\cdot\Delta^n 1^s \cdot \frac{\beta_{k-s}^{(k)}(1)}{(k-s)!},$$

where $\Delta^n 1^s = \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}(r+1)^s$, and where the series converges for small $q-1$. He also obtains a wide variety of related expansion formulas. Next he makes a similar study of q-Eulerian polynomials $\varepsilon_n^{(z)}(x, q)$ of unrestricted order z. For $z=1$ many of the results reduce to formulas in one of his earlier papers [same J. **15** (1948), 987–1000; MR **10**, 283]. In the latter half of the paper the author derives a number of formulas which relate his rational function $H_n(x, q)$ [Trans. Amer. Math. Soc. **76** (1954), 332–350; MR **15**, 686] to $\beta_n(0, q)$ and $\varepsilon_n(0, q)$. Sharma [#2479 above] has generalized the polynomial $\beta_n(x, q)$ in a different manner. The author's comprehensive generalization, formulated in conclusion, includes all previous definitions.

A. L. Whiteman (Los Angeles, Calif.)

2481:
Griffith, James L. On the zeros of a certain function involving Bessel functions. J. Proc. Roy. Soc. New South Wales **91** (1957), 190–196.

The author proves a number of results concerning the location of the zeros of

$$w(z) = zH_{\nu+1}^{(1)}(z) - CH_\nu^{(1)}(z),$$

where $\nu \geqq 0$, $-\frac{1}{2}\pi \leqq \arg z \leqq \frac{3}{2}\pi$, and C is a real constant. Some of these results were stated without proof and used in an earlier article by the author [J. Proc. Roy. Soc. New South Wales **90** (1956), 190–196; MR **19**, 650].

Typical results are that the zeros are symmetrically placed with respect to the imaginary axis; that any multiple zero lies on one of the coordinate axes; that if $C < 2\nu$, $w(z)$ has no zeros in the upper half plane, and if $C > 2\nu$, $w(z)$ has but one zero there.

Except in certain exceptional cases, the method of proof is the application of the argument principle to $w(z)/zH_\nu^{(1)}(z)$. *P. G. Rooney* (Toronto, Ont.)

2482:
Moran, P. A. P. Inequalities for the Bessel function $I_n(x)$. Quart. J. Math. Oxford Ser. (2) **8** (1957), 287–290.

If we approximate the integral representation $(2\pi)^{-1}\int_0^{2\pi}\exp(x\cos\theta)d\theta$ for the modified Bessel function $I_0(x)$ by a sum taken over m equally-spaced points, we obtain

$$\frac{1}{m}\sum_{s=1}^{m}\exp\left\{x\cos\left(\delta + \frac{2\pi s}{m}\right)\right\} = S(\delta),$$

say, where δ is arbitrary. The author proves that

$$S(\pi/m) \leqq I_0(x) \leqq S(0) \quad (m \text{ even}),$$

together with an extension of this result to $I_n(x)$ ($n=1, 2, \cdots$). The bounds are illustrated by numerical examples. *F. W. J. Olver* (Teddington)

2483:
Henrici, Peter. On the representation of a certain integral involving Bessel functions by hypergeometric series. J. Math. Phys. **36** (1957), 151–156.

The integral $g(z) = \int_0^\infty J_\mu(at)J_\nu(bt)K_\lambda(zt)dt$ involving Bessel and modified Hankel functions is expressed in the form of a product of two hypergeometric series. From this an expression for

$$f(a, b; c) = \int_0^\infty J_\mu(at)J_\nu(bt)J_\lambda(ct)dt$$

is derived. This formula is dicussed for arbitrary positive a, b, c (previous investigations of $f(a, b, c)$ by Bailey were restricted to $c > a+b$). Special values of the parameters ($a=b=c$ and $\mu=\nu=\lambda$ or $\mu=\nu=-\lambda$) lead to results expressed in terms of the Gamma function.

F. Oberhettinger (Madison, Wis.)

2484:
Vilenkin, N. Ya. Some relations for Gegenbauer functions. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 3(81), 167–172. (Russian)

A number of integral relations and sums of products are derived, each of which has a group theoretic interpretation. For example,

$$C_n^p(\cos\lambda) = \frac{2\Gamma(2p+n)\Gamma(p+\frac{1}{2})\Gamma(2t)}{\Gamma(2p)\Gamma(p-t)\Gamma(n+2t)\Gamma(t+\frac{1}{2})}$$

$$\times\int_0^{\frac{\pi}{2}} C_n^t(\cos\lambda\Theta^{-\frac{1}{2}})\Theta^{n/2}\sin^{2t}\theta\cos^{2p-2t-1}(\theta)d\theta,$$

where $\Theta = \sin^2\theta + \cos^2\lambda\cos^2\theta$, if $\Re p > \Re t > -\frac{1}{2}$. This formula is connected with the decomposition, into a continuous direct sum of irreducible unitary representations, of the representation of the rotation group of $E^{(2t+1)}$ induced by an irreducible unitary representation of the rotation group of $E^{(2p+1)}$.

N. D. Kazarinoff (Ann Arbor, Mich.)

2485:

Arscott, F. M. Integral equations for ellipsoidal wave functions. Quart J. Math. Oxford Ser. (2) 8 (1957), 223–235.

An elliptical wave function $w = \mathrm{el}\, z$ is by definition a solution of the differential equation

$$\frac{d^2w}{dz^2} - [a + bk^2 \operatorname{sn}^2 z + qk^4 \operatorname{sn}^4 z]w = 0,$$

which is uniform and doubly periodic in z, with periods $2K$ or $4K$, $2iK'$ or $4iK'$ (notations $\operatorname{sn} z$, k, K, K' as usual in the theory of elliptic functions). Two theorems concerning integral equations of Möglich's type

$$\mathrm{el}\,\alpha\,\mathrm{el}\,\beta = \lambda i \iint_S (\operatorname{sn}^2 \alpha' - \operatorname{sn}^2 \beta') F(\alpha, \beta; \alpha', \beta')\,\mathrm{el}\,\alpha'\,\mathrm{el}\,\beta'\, d\alpha' d\beta'$$

and of Malurkar's type

$$\mathrm{el}\,\alpha = \lambda i \iint_S (\operatorname{sn}^2 \beta - \operatorname{sn}^2 \gamma) f(\alpha, \beta, \gamma)\,\mathrm{el}\,\beta\,\mathrm{el}\,\gamma\, d\beta d\gamma$$

are formulated and proved. Numerous kernels $F(\alpha, \beta, \alpha', \beta')$ and $f(\alpha, \beta, \gamma)$ are explicitly given; they can be obtained as solutions of certain partial differential equations. By specializing α or β in Möglich's equation, one arrives at an integral equation of Malurkar's type; but it seems that not all integral equations of this type can be derived in this way from Möglich's type. *J. Meixner* (Aachen)

2486:

Arscott, F. M. Relations between spherical and ellipsoidal harmonics and some applications. J. London Math. Soc. 33 (1958), 39–49.

Die Laméschen Polynome $E_n{}^p(\alpha)$ lassen sich in der Gestalt $\operatorname{sn}^r \alpha \operatorname{cn}^s \alpha \operatorname{dn}^t \alpha F(\operatorname{sn}^2 \alpha)$ mit $r, s, t = 0$ oder 1 und einem Polynom F schreiben. Es gibt sonach acht Klassen Laméscher Polynome. Die Produkte $E_n{}^p(\alpha)E_n{}^p(\beta)$ für diese acht Klassen entsprechen acht Klassen von Kugelflächenfunktionen

$$P_n{}^m(\cos\theta)\cos m\varphi \text{ und } P_n{}^m(\cos\theta)\sin m\varphi.$$

Die Entwicklung eines solchen Produktes nach der entsprechenden Klasse von Kugelflächenfunktionen wird untersucht. Ferner werden gewisse Kerne $K(\alpha, \beta, \gamma)$, die in Integralgleichungen für Ellipsoidwellenfunktionen vom Malurkar'schen Typ auftreten [F. M. Arscott, #2485 obenstehend] nach Produkten einer Bessel-Funktion und zweier Laméscher Polynome entwickelt.
J. Meixner (Aachen)

2487:

Macrobert, T. M. Integrals involving hypergeometric functions and E-functions. Proc. Glasgow Math. Assoc. 3 (1958), 196–198.

Several definite integrals involving products of a ${}_2F_1$ and an E-function are evaluated.
N. D. Kazarinoff (Ann Arbor, Mich.)

2488:

Ragab, F. M. Transcendental addition theorems for the hypergeometric function of Gauss. Pacific J. Math. 8 (1958), 141–145.

Integrals involving products of two ${}_2F_1$'s, regarded as functions of their parameters, are evaluated in terms of other ${}_2F_1$'s. *N. D. Kazarinoff* (Ann Arbor, Mich.)

2489:

Ragab, F. M. On the product of two confluent hypergeometric functions. Arch. Math. 8 (1957), 180–183.

Confluent forms of some integrals involving products of hypergeometric functions by the same author [#2488 above]. *P. Henrici* (Los Angeles, Calif.)

2490:

Henrici, Peter. On the product of two Kummer series. Canad. J. Math. 10 (1958), 463–467.

Considérant le produit

$$(1)\quad \Phi(z) = {}_1F_1\left[\begin{matrix}\mu+\frac{1}{2}-\alpha;\ -z\\ 2\mu+1\end{matrix}\right]{}_1F_1\left[\begin{matrix}\nu+\frac{1}{2}-\beta;\ z\\ 2\nu+1\end{matrix}\right],$$

dont l'invariance par le changement de $(\alpha, \beta, \mu, \nu; z)$ en $(\beta, \alpha, \nu, \mu; -z)$ est évidente, l'auteur applique la transformation de Kummer aux deux séries de (1) et obtient ainsi

$$(2)\quad \Phi(z) = {}_1F_1\left[\begin{matrix}\mu+\frac{1}{2}+\alpha;\ z\\ 2\mu+1\end{matrix}\right]{}_1F_1\left[\begin{matrix}\nu+\frac{1}{2}+\beta;\ -z\\ 2\nu+1\end{matrix}\right],$$

montrant ainsi l'invariance, non triviale, de $\Phi(z)$ par le changement

$$(\alpha, \beta, \mu, \nu; z) \text{ en } (-\alpha, -\beta, \mu, \nu; -z).$$

Il en résulte que si l'on pose $\Phi(z) = \sum_0^\infty a_n z^n$, on doit avoir pour les a_n les propriétés:

$$(3)\quad a_n(\beta, \alpha, \nu, \mu) = a_n(-\alpha, -\beta, \mu, \nu) = (-1)^n a_n(\alpha, \beta, \mu, \nu).$$

L'auteur en déduit une fonction génératrice pour les coefficients a_n mettant en évidence la propriété exprimée par (3): plus généralement, il obtient un ensemble complet de fonctions génératrices de forme

$$\psi(z) = \sum_0^\infty c_n a_n z^n,$$

$\psi(z)$ étant le produit de deux fonctions hypergéométriques généralisées. Une application est donnée, pour terminer, au produit de 2 fonctions de Whittaker.
R. Campbell (Caen)

2491:

Campbell, Robert. Expression asymptotique des sommes de Fejer pour les développements de polynomes orthogonaux classiques. C. R. Acad. Sci. Paris 246 (1958), 1647–1648.

L'auteur obtient l'expression asymptotique du noyau des sommes de Fejer pour les séries qui sont les développements de fonctions en séries de polynomes de Jacobi.
N. D. Kazarinoff (Ann Arbor, Mich.)

2492:

Proriol, Joseph. Sur une famille de polynomes à deux variables orthogonaux dans un triangle. C. R. Acad. Sci. Paris 245 (1957), 2459–2461.

The author constructs a (doubly-indexed) sequence of polynomials $\mathfrak{P}_{m,n}(x, y)$ which are orthogonal over the triangle $x \geqq 0$, $y \geqq 0$, $x+y \leqq 1$, with the weight function

$$[1-(\tau/\rho)]^{\gamma-1}[1+(\tau/\rho)]^{\gamma'-1}(2-2\rho)^{-\gamma'-\gamma+\alpha}(2\rho)^{\gamma+\gamma'-1},$$

where $2x = \rho - \tau$, $2y = \rho + \tau$, $\operatorname{Re}(\gamma) > 0$, $\operatorname{Re}(\gamma') > 0$, $\operatorname{Re}(\alpha) > \operatorname{Re}(\gamma+\gamma') - 1$, and m is the total degree of $\mathfrak{P}_{m,n}$. The polynomials are explicitly exhibited as products of two Jacobi polynomials in τ/ρ and $2\rho - 1$, respectively, are eigenfunctions of two commuting partial differential operators of second order, and are orthogonal in each index separately, unlike the comparable generalization of Jacobi polynomials due to Appel and Kampé de Fériet [cf. A. Erdélyi et al., Higher transcendental functions, McGraw-Hill, New York, v. I, pp. 244, v. II, p. 259; MR 15, 419]. For a special choice of γ, γ', α, Proriol polynomials coincide with those recently encountered in con-

nection with the Schrödinger equation for helium [cf. Munschy et Pluvinage, #3011 below].

A. B. Novikoff (Berkeley, Calif.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 2638, 2640, 2784, 2876, 2877, 2973.

2493:

Bellman, Richard. On monotone convergence to solutions of $u'=g(u,t)$. Proc. Amer. Math. Soc. **8** (1957), 1007–1009.

The author proves that if $g(u,t)$ is continuously differentiable and convex in u for $t\in[0,t_0]$ and if $v(t)$ is an arbitrary continuous function there, then the sequence $\{u_n(t)\}$ defined by $u_0'(t)=g(v,t)+(u_0-v)g_u(v,t)$, $u_{n+1}'(t)=g(u_n,t)+(u_{n+1}-u_n)g_u(u_n,t)$ $(u'=du/dt,\ g_u=\partial g/\partial u)$, $u_n(0)=c$, $n=0,1,2,\cdots$, is non-decreasing in n for $t\in[0\,t_1]$. This sequence is also uniformly bounded and its limit is a solution of the differential equation $u'=g(u,t)$. The clearly written proof is partly sketched, partly given in extenso. Also, the possibility of extension of the result to systems of differential equations is examined.

J. Aczél (Debrecen)

2494:

Consiglio, Alfonso. Sulla canonicità di un sistema di equazioni differenziali e sua equivalenza ad un sistema lagrangiano. Matematiche, Catania **11** (1956), 135–143 (1957).

The paper is concerned with necessary and sufficient conditions that a given system of $2n$ differential equations can be written in Hamiltonian or Lagrangian form, using no other transformation than that of Hamilton. The general formalism is therefore well known; in particular what the author describes as "una certa reciprocità" between the Hamiltonian and Lagrangian functions.

D. C. Lewis, Jr. (Baltimore, Md.)

2495:

Hukuhara, Masuo. Sur la relation de Fuchs relative à l'équation différentielle linéaire. Proc. Japan Acad. **34** (1958), 102–106.

In order to extend the formula of Fuchs to the case of a linear differential equation that admits irregular singular points, the author first considers the linear differential equation

$$(*)\quad x^{(m+1)n}y^{(n)}+x^{(m+1)(n-1)}p_1(x)y^{(n-1)}+\cdots +x^{m+1}p_{n-1}(x)y'+p_n(x)y=0,$$

where $p_1(x),\cdots,p_n(x)$ are analytic at $x=0$, as the limit of the equation

$$(**)\quad (x^{m+1}-a^{m+1})^n y^{(n)}+(x^{m+1}-a^{m+1})^{n-1}p_1(x)y^{(n-1)}+\cdots +(x^{m+1}-a^{m+1})p_{n-1}(x)y'+p_n(x)y=0$$

as $a\to 0$. The equation

$$\alpha^n p_1(0)\alpha^{n-1}+\cdots+p_{n-1}(0)\alpha+p_n(0)=0,$$

determined by (*) at $x=0$, is assumed to admit n distinct roots α_{0k}, $k=1,\cdots,n$, so that the origin is an ordinary singular point of multiplicity $m+1$ of the equation (*). With $\lambda_{j1},\cdots,\lambda_{jn}$ denoting the roots of the equation

$$\lambda(\lambda-1)\cdots(\lambda-n+1)+\frac{1}{m+1}\,\frac{p_1(\varepsilon^j a)}{(\varepsilon^j a)^m}\times\lambda(\lambda-1)\cdots(\lambda-n+2)+\cdots=0,$$

determined by (**) at $x=\varepsilon^j a$, where $\varepsilon=\exp\{2\pi i/(m+1)\}$, $j=0,1,\cdots,m$, the basic lemma is established that

$$\lim_{a\to 0}\sum_{j=0}^{m}\sum_{k=1}^{n}\lambda_{jk}=\sum_{k=1}^{n}\lambda_k,$$

where the λ_k are the exponential values λ in the n formal solutions of (*), which are solutions of the form

$$y=e^{A(x)}x^\lambda\{y_0+y_1x+\cdots+y_mx^m+\cdots\},$$

with

$$A(x)=-\frac{\alpha_0}{mx^m}-\frac{\alpha_1}{(m-1)x^{m-1}}-\cdots-\frac{\alpha_{m-1}}{x},$$

corresponding to the above n values α_{0k} of α_0.

Now consider the equation

$$(***)\quad q_0(x)y^{(n)}+q_1(x)y^{(n-1)}+\cdots+q_{n-1}(x)y'+q_n(x)y=0,$$

where $q_0(x),\cdots,q_n(x)$ are polynomials in x. The zeros $a_1,\cdots,a_m$ of $q_0(x)$, and the point at ∞, are the singular points of (***). If a_j is a regular singular point (i.e., an ordinary singular point of multiplicity $\mu_j=1$), then the values $\lambda_{j1},\cdots,\lambda_{jn}$ are taken to be the n roots of the equation determined by (***) at $x=a_j$. If a_j is an ordinary singular point of multiplicity $\mu_j>1$, then the values $\lambda_{j1},\cdots,\lambda_{jn}$ are taken to be the exponents in the solutions. The point at ∞, considered by means of the transformations $x=1/t$, is treated similarly. Suppose that $a_1,\cdots,a_m,\infty$ are regular singular points or ordinary multiple singular points. Since (***) is the limit of an equation of the type of Fuchs admitting $M=\mu_1+\cdots+\mu_m+\mu_\infty$ regular singular points, the generalized formula of Fuchs,

$$\sum_{k=1}^{n}\lambda_{\infty k}+\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_{jk}=(M-2)n(n-1)/2,$$

now follows readily from the result given in the preceding paragraph.

E. F. Beckenbach (Los Angeles, Calif.)

2496:

Haimov, N. B. Application of the Briot-Bouquet equation to the investigation of the general case. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **21** (1957), 113–122. (Russian)

This is a detailed study of the nature of the paths near the origin of

$$\frac{dy}{dx}=\frac{Y_n(x,y)+\cdots}{X_n(x,y)+\cdots},$$

where the numerators and denominators are power series whose lowest degree terms are X_n, Y_n (of degree n). There are earlier writings of the author on this topic [Uč. Zap. Stalinabad. Gos. Univ. **1**, 2 (1952)]. He returns here to these questions particularly to determine whether a certain direction of approach is reached by one or by an infinity of paths. The general method is by a repeated application of a transformation of type $y_1=y/x$ and a reduction to the Briot-Bouquet form

$$\frac{dy}{dx}=\frac{f_n(y)+xg(x,y)}{x},$$

where f_n is a series beginning with y^n.

S. Lefschetz (Princeton, N.J.)

2497:

Kukles, I. S.; and Gruz, D. M. On the number of operations connected with the application of the Frommer method. Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. **1958**, no. 1, 29–45. (Russian. Uzbek summary)

A very complicated form of $dy/dx=Y(x,y)/X(x,y)$ (X, Y power series) is taken. A no less complicated type of

operation is applied and theorems are given regarding the required number of such operations to reduce the origin as a critical point. [References: Kukles, Izv. Akad. Nauk Uz SSR **1957**, no. 4, 85–95; Haimov, Uč. Zap. Stalinabad. Ped. Inst. **1952**, no. 1; Gruz, Trudy Uzbek. Gos. Univ. (1957) (in press)]. *S. Lefschetz* (Princeton, N.J.)

2498:

Babister, A. W. Determination of the optimum response of linear systems (zero displacement error systems). Quart. J. Mech. Appl. Math. **10** (1957), 504–512.

The author analyses the performance of linear systems for which the equation of motion is

$$(1)\quad a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = f(t),$$

where the a_i are constants with $a_n \neq 0$. Two response functions, $L = \int_0^\infty e^2 dt$ and $L_1 = \int_0^\infty (de/dt)^2 dt$, previously defined [Quart. J. Mech. Appl. Math. **10** (1957), 360–368; MR **19**, 415] are used to analyse the performance of systems for which L is near its minimum while L_1 is considerably less than its value at $L_{\min}$. It is found that the overshoot and L_1 are correlated in such a manner that the smaller L_1, the smoother the response.

Values of $L_{\min}$ and the coefficients in (1) for zero-displacement-error systems are given in tabular form. An approximate method shows how the response functions may be used to get optimum response for such systems. *C. G. Maple* (Ames, Iowa)

2499:

Babister, A. W. Determination of the optimum response of linear systems (zero-velocity-error and zero-acceleration-error systems). Quart. J. Mech. Appl. Math. **11** (1958), 119–128.

The work of two previous papers [Quart. J. Mech. Appl. Math. **10** (1957), 360–368; MR **19**, 415; #2498 above] is extended to zero-velocity-error and zero-acceleration-error systems. *C. G. Maple* (Ames, Iowa)

2500:

Kostomarov, D. P. Formal systems of linear differential equations and their solutions in the form of normal and subnormal series. Mat. Sb. N.S. **44(86)** (1958), 137–156. (Russian)

The author investigates the solutions of systems of linear differential equations of the form $w_i^1 = \sum_{j=1}^n a_{ij}(z) w_j$ ($j = 1, 2, \cdots, n$), where $a_{ij}(z) = \sum_{m=0}^\infty a_{ij}^{(m)} z^{(s-m)/q}$ ($a_{ij}^{(m)}$ are constants, s, q are integers, $q > 0$); the solutions to have the form of normal and subnormal series as introduced by Poincaré.

He shows that such systems of equations always have a fundamental system of normal and subnormal solutions. He indicates the method for constructing such solutions and studies their properties.

S. D. Zeldin (Cambridge, Mass.)

2501:

Fadini, Angelo. Riducibilità alle quadrature di sistemi differenziali lineari omogenei dovuta a speciali simmetrie della matrice dei coefficienti. I, II. Ricerca, Napoli (2) **8** (1957), Gennaio-Giugno, 42–59.

The author is concerned with the reduction of an nth order vector equation $d\bar{y}/dx = M\bar{y}$ to a new equation $d\bar{z}/dx = D\bar{z}$ by means of a non-singular transformation $\bar{y} = P\bar{z}$ so as to give D either a diagonal form, or at least a block diagonal form. The elements of P are to be constants, while those of M may vary with x. Such reductions are usually possible only when special symmetries appear in the structure of M. The author has discovered a new and broader class of equations where such reductions are possible.

For example, if $n = 3$, a_1, a_2, a_3 are arbitrary constants; and, if u_0, u_1, and u_2 are three arbitrary integrable functions of x, and

$$M = \begin{Vmatrix} u_0 & a_1 u_1 & a_1 a_2 u_2 \\ a_2 a_3 u_2 & u_0 & a_2 u_1 \\ a_3 u_1 & a_3 a_1 u_2 & u_0 \end{Vmatrix},$$

it is shown that reduction to diagonal form is possible and the general solution of the given differential system is given in terms of the quadratures of the u's. In part I the nth order case, analogous to the special third order problem here indicated, is carried out in detail with full diagonalization.

In part II the results of part I are extended by making all the u's square matrices of the same order which vary with x, rather than letting them be just scalar functions of x. When the u's are matrices, block diagonalization is obtained and no further diagonalization of the type contemplated is in general possible without assuming further symmetries in the u matrices. Such a possible symmetry involving cyclic matrices is considered and full diagonalization is obtained.

H. L. Turrittin (Minneapolis, Minn.)

2502:

Fadini, Angelo. Sull'integrazione di un sistema di equazioni differenziali lineari omogenee la cui matrice dei coefficienti è composta mediante matrici circolanti. Ricerca, Napoli (2) **8** (1957), Luglio-Dicembre, 17–31.

The vector differential equation

$$(1)\qquad \frac{d\bar{y}}{dx} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix} \bar{y},$$

in which the elements of the square matrices

$$A_i = \begin{pmatrix} a_i & b_i & c_i \\ c_i & a_i & b_i \\ b_i & c_i & a_i \end{pmatrix} \ (i = 1, 2)$$

are integrable functions of x, is reduced by a judiciously chosen non-singular transformation $\bar{y} = P\bar{z}$ to a new equation $d\bar{z}/dx = B\bar{z}$, where B is in diagonal form. Thus it is shown that the integration of (1) can be reduced to quadratures and in terms of these quadratures the general solution of (1) is given.

H. L. Turrittin (Minneapolis, Minn.)

2503:

van Kampen, N. G. Spectral decomposition of the operator $p^2 - q^2$. Physica **24** (1958), 545–556.

A complete set of eigenfunctions $\varphi_\lambda(x)$ for the equation $(-d^2/dx^2 - x^2)y = \lambda y$ is given in terms of Weber's function $D_\nu(z)$, with $\nu = -\frac{1}{2} - \frac{1}{2}i\lambda$ and $z = \pm(1+i)x$. This equation occurs in the theory of a non-relativistic electron in an electromagnetic field. An initial error by van Kampen [Danske Vid. Selsk. Mat.-Fys. Medd. **26** (1951), no. 15; MR **13**, 807], repeated by Enz [Nuovo Cimento (10) **3** (1956), supplemento, 363–417; MR **19**, 364] and by Arnous [J. Phys. Rad. **17** (1956), 374–375], is thus corrected, in agreement with the work of Steinwedel [Fortschritte der Physik **1** (1953), 7–28]. In appendices, some properties of φ_λ are obtained, including asymptotic behavior for large x and large λ.

K. Wildermuth and K. Baumann [Nuclear Physics **3** (1957), 612–623] used the φ_λ to study the evolution of wave-packets. Without explicit use of the eigenfunctions, van Kampen shows that the centre of gravity of the wave-

packet moves like the classical self-accelerated electron, and that the wave-packet always spreads out.

A. J. Coleman (Toronto, Ont.)

2504:

Bellman, Richard. On the determination of characteristic values for a class of Sturm-Liouville problems. Illinois J. Math. 2 (1958), 577–585.

The paper considers the problem of determining the characteristic values $\lambda_1<\lambda_2<\cdots$ of the Sturm-Liouville problem $u''+\lambda a(t)u=0$, where $0<a^2\leq a(t)\leq b^2<\infty$ on $0\leq t\leq 1$, and $u(0)=u(1)=0$. A method involving sequences of upper and lower bounds is given for calculating, by hand or digital computer, the least characteristic value λ_1, in particular, and also products $\lambda_1\lambda_2\cdots\lambda_k$. The method is applied to the boundary value problem in which $a(t)=1+t$. The special case in which $a(t)$ is a polynomial is discussed further. Finally, an extension of the method to the higher order equation $u^{(4)}+\lambda a(t)u=0$, for which $u(0)=u'(0)=u(1)=u'(1)=0$, is considered.

C. R. Putnam (Lafayette, Ind.)

2505:

Hille, Einar. Über eine Klasse Differentialoperatoren vierter Ordnung. S.-B. Berlin. Math. Ges. **1954/55–1955/56**, 39–44.

Let $b(x)$ be continuous and satisfy $b(x)>M(1+|x|)^{4+\alpha}$, $\alpha>1$, in $(-\infty,\infty)$ with a positive constant M. Then it is proved that the equation $L_4(y)-\lambda y=0$, where $L_4(y)=-(b(x)y'')''$, admits negative eigenvalues λ_n in $L(-\infty,\infty)$ such that the corresponding eigenfunctions $\omega_n(x)$ are closed and complete in $L(-\infty,\infty)$, as well as in $L_2(-\infty,\infty)$. Moreover, the absolutely and uniformly convergent series $G(x,s,\lambda)=\sum_n(\lambda-\lambda_n)^{-1}\omega_n(x)\omega_n(s)$ give, for $\lambda>0$, the kernel of the resolvent in $L(-\infty,\infty)$ of the operator L_4, so that the corresponding semi-group T_t in $L(-\infty,\infty)$ with the infinitesimal generator L_4 given by $T_tf(x)=\sum_n f_n\omega_n(x)\exp(\lambda_n t)$, with $f(x)\sim\sum_n f_n\omega_n(x)$, is not only strongly continuous for $t\geq 0$ but is also analytic for $\mathrm{Re}(t)>0$. These results are obtained by generalizing the author's method [J. Analyse Math. 3 (1954), 81–196; MR **16**, 45] of proving that $L_2(y)=(b(x)y')'$ generates, in $L(-\infty,\infty)$, a strongly continuous semi-group, if $b(x)$ is $>M(1+|x|)^{2+\alpha}$, $\alpha>0$.

K. Yosida (Tokyo)

2506:

Krasnosel'skiĭ, M. A. On a boundary problem. Izv. Akad. Nauk. SSSR. Ser. Mat. **20** (1956), 241–252. (Russian)

The non-linear boundary problem

$$y''+f(x,y,y')=0,\ y(0)=y(\pi)=0$$

is considered, where the function $f(x,y,v)$ is continuous in the strip $0\leq x\leq\pi$, $-\infty<y$, $v<\infty$ and satisfies the conditions

$$f(x,y,v)\leq\sigma_1|y|+\sigma_2|v|+\psi(x)\ \text{for}\ y\geq 0;$$
$$f(x,y,v)\geq-\sigma_1|y|-\sigma_2|v|-\psi(x)\ \text{for}\ y\leq 0,$$

with $\psi(x)\in L[0,\pi]$, $\sigma_1\geq 0$, $\sigma_2\geq 0$ and $\sigma_1+\sigma_2<1$. It is proved that this problem has at least one solution. In the case that the identically vanishing function is a solution, sufficient conditions are given for the existence of a second solution. The proofs use the methods of functional analysis. The author's theorems generalize the corresponding result of Tonelli [Ann. Scuola Norm. Sup. Pisa **8** (1939), no. 2, 75–88].

B. P. Demidovič (RŽMat **1957** 2309)

2507:

Kasuga, Takashi. On eigenfunction expansions of self-adjoint ordinary differential operators. I, II. Proc. Japan Acad. 33 (1957), 591–595; 34 (1958), 38–43.

The author considers the second-order formally self-adjoint operator

$$\tau=-\frac{d}{dt}\left(p(t)\frac{d}{dt}\right)+q(t)\ (a<t<b),$$

and an open subset U of its spectrum within which the self-adjoint operators derivable from the restriction $\tau|[c,b)$ of τ to a half-closed subinterval of (a,b) have a discrete spectrum. Let T be a given self-adjoint operator obtained from τ by the imposition of a suitable number of boundary conditions at each end point a, b. Let $\{\rho_{jk}(e)\}$ $(j,k=1,2)$ be the positive matrix-valued set function in terms of which the spectral representation of T is established [as in the well-known paper of Kodaira, Amer. J. Math. **71** (1949), 921–945; MR **11**, 438]. Let $\rho(e)=\rho_{11}(e)+\rho_{22}(e)$. It is shown that for subsets e of U one has

$$\rho_{ij}(e)=\int_e\theta_i(\lambda)\theta_j(\lambda)\rho(d\lambda)\ (i,j=1,2),$$

explicit expressions for the functions θ_i being given. It follows that, within U, the spectrum of T is simple. Various applications and related results are also given.

J. Schwartz (New York, N.Y.)

2508:

Greguš, Michal. Über einige Zusammenhänge zwischen der Integralen der gegenseitig adjungierten linearen Differentialgleichungen der dritten Ordnung und über ein Randwertproblem. Acta Fac. Nat. Univ. Comenian. Math. **1** (1956), 265–272. (Slovak. Russian and German summaries)

2509:

Greguš, M. Über die lineare Differentialgleichung der dritten Ordnung mit konstanten Koeffizienten. Acta Fac. Nat. Univ. Comenian. Math. **2** (1957), 61–66. (Slovak. Russian and German summaries)

In dieser Arbeit behandelt man einige Eigenschaften der Lösungen der linearen Differentialgleichung dritter Ordnung mit konstanten Koeffizienten der Form:

(a) $$y'''+2Ay'+\Omega y=0$$

und die Eigenschaften der Lösungen der zu ihr adjungierten:

(b) $$z'''+2Az'-\Omega z=0,$$

dabei sind $A>0$, $\Omega>0$ Konstanten.

Im zweiten Teil wird die Existenz der Eigenwerten für die Differentialgleichung (a) und für die Randwertaufgabe

$$y(x_0-d,\lambda)=y(x_0,\lambda)=y''(x_0,\lambda)=0$$

durchgeführt, wo $x_0\in(-\infty,\infty)$, $d>0$ konstante Zahlen sind. Dabei $A=A(\lambda)$, $\Omega=\Omega(\lambda)$ bedeuten für $\lambda\in(\Lambda_1,\Lambda_2)$ stetige Funktionen mit bestimmten Eigenschaften.

Zusammenfassung des Autors

2510:

Štraus, A. V. On generalized resolvents and spectral functions of differential operators of even order. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 785–808. (Russian)

A selfadjoint real ordinary differential operator of order $2n$ over (a,b) is written in canonical form $l(y)=\sum(-1)^r(d/dx)^{n-r}(p_ry)$, where $p_0^{-1}, p_1,\cdots,p_n$ are measurable in (a,b) and summable over each $[\alpha,\beta]\subset(a,b)$. D is the set of y in $L^2(a,b)$ for which $l(y)\in L^2(a,b)$, L the

operator with minimal domain D_L consisting of those y in D such that $(l(y), z)=(y, l(z))$ for all z in D, $L(y)=l(y)$ in D_L. For all nonreal λ, N_λ is the nullmanifold of $L^*-\lambda$, m the dimension of N_λ. Let F be any linear mapping of N_{λ_0} in $N_{\bar\lambda_0}$. A quasi-selfadjoint extension L_F of L is defined as the part of L^* with domain $D_{L_F}=\{g: g=f+F\varphi-\varphi, f\in D_L, \varphi\in N_{\lambda_0}\}$. Then $L_F^*=L_{F^*}$; the domains of these operators are characterised by boundary conditions, equivalent to those given.

The generalised resolvent of L, $R_\lambda=(L_{F(\lambda)}-\lambda E)^{-1}$ where $\operatorname{Im}\lambda\cdot\operatorname{Im}\lambda_0>0$, and $\|F(\lambda)\|\leqq 1$, is now calculated explicitly in the form $R_\lambda f=\int K(x, s; \lambda)f(s)ds$, where K is given in terms of a fundamental system of solutions of $l(y)=0$ and is of summable square in x (or s) over (a, b) for each s (or x), and $K(x, s; \bar\lambda)=\overline{K(s, x; \lambda)}$.

If $y(x, \lambda)$ is a $2n$ column vector, whose elements form a fundamental system of solutions of $l(y)=y$, normalised by prescribed rules, then

$$K(x, s, \lambda)=y'(x, \lambda)[M(\lambda)+\tfrac{1}{2}\operatorname{sgn}(x-s)J]y(s, \lambda),$$

where $M(\lambda)$ is a $2n\times 2n$ matrix defined in terms of a fundamental system of solutions of $l(y)=0$, $J_{rs}=0$ if $r+s\neq 2n$, $J_{r,2n-r}=\mp 1$ as $r\leqq n$ or $r>n$. $M(\lambda)$ is called the characteristic matrix of R_λ. $\operatorname{Im} M=(M-M^*)/2i\geqq 0$ in the upper halfplane, and by means of an extension to matrices of the theorem on representation of functions with positive real parts by a Stieltjes integral it is shown that $R_\lambda=\int (t-\lambda)^{-1}dE_t$ for a spectral function E_t. Explicit formulae are given:

$$(E_\beta-E_\alpha)f=\int_a^b K(x, s; \alpha, \beta)f(s)ds,$$

where $K(x, s; \alpha, \beta)=\int_\beta^\alpha y'(x, \sigma)dT(\sigma)y(s, \sigma)$, $\pi T(\sigma)=\lim_{\tau\to 0}\int_0^\sigma \operatorname{Im} M(\sigma+i\tau)d\sigma$. K is of summable square over (a, b) in each of x and s. *J. L. B. Cooper* (Cardiff)

2511:

Stampacchia, Guido. Problemi ai limiti per i sistemi di equazioni differenziali ordinarie. Matematiche, Catania **11** (1956), 121–134 (1957).

After a brief survey of literature on generalizations of the problems of Picard and Nicoletti, the author discusses the topological method of solution of boundary problems that he has introduced earlier [Ricerche Mat. **3** (1954), 76–94; MR **16**, 363]. Specifically, he considers here the three-dimensional problem of determining a solution of the differential system $y_i'=f_i(x, y_1, y_2)$ $(i=1, 2)$ that passes through two given curves V_1, V_2, where in the three-dimensional strip C: $a\leqq x\leqq b$, $|y_i|<\infty$ $(i=1, 2)$, the functions $f_i(x, y_1, y_2)$ are continuous in (x, y_1, y_2) and Lipschitzian in (y_1, y_2) ,the curves V_1, V_2 lie in C, while there exist summable functions $\varphi_{ij}(x)$, $\phi_i(x)\geqq 0$, $a\leqq x\leqq b$, such that $|f_i(x, y_1, y_2)-\sum_{j=1}^2 \varphi_{ij}(x)y_j|\leqq \phi_i(x)$, $i=1, 2$, on C. *W. T. Reid* (Evanston, Ill.)

2512:

Ráb, Miloš. Asymptotische Eigenschaften der Lösungen linearer Differentialgleichung dritter Ordnung. Publ. Fac. Sci. Univ. Masaryk **1956**, 177–184. (Russian summary)

Properties of solutions of the canonical equation

$$(*)\qquad y'''+4A(x)y'+[2A'+\omega(x)]y=0$$

are compared with those of solutions of

$$(**)\qquad y''+A(x)y=0.$$

If all solutions of (**) are bounded on (x_0, ∞), all solutions of (*) are. If all solutions of (**) converge to zero as $x\to\infty$ and $\int_{x_0}^\infty |\omega|dt<\infty$, then all solutions of (*) have limit zero as $x\to+\infty$. If on (x_0, ∞) it is true that 1) $A(x)>0$ and $\lim_{x\to\infty} A(x)=\infty$, 2) $A^{-\frac14}(x)$ is convex, $\int_{x_0}^\infty |\omega/A|dt<\infty$, then each solution of (*) is $O(A^{-\frac14})$ and has a bounded first derivative on (x_0, ∞).

N. D. Kazarinoff (Ann Arbor, Mich.)

2513:

Ráb, Miloš. Asymptotische Eigenschaften der Lösungen linearer Differentialgleichung dritter Ordnung. Publ. Fac. Sci. Univ. Masaryk **1956**, 441–454. (Czech. Russian and German summaries)

The canonical equation

$$(*)\qquad y'''-a(x)y'-[\omega(x)+a'(x)/2]y=0$$

is considered. If $\lim_{x\to\infty} a(x)=c>0$, $\int^\infty |da(x)|<\infty$, and $\int^\infty |\omega|dx<\infty$, then (*) has a fundamental system of solutions y_i $(i=1, 2, 3)$ such that $y_1=\exp[-\int^x a^{\frac12}dt][1+o(1)]$, $y_2=1+o(1)$, and $y_3=\exp[\int^x a^{\frac12}dt][1+o(1)]$. If $a(x)=1+h(x)$, where $\int^x |h|dx<\infty$, then the same conclusion holds. If $a>0$, $\int^\infty a^{\frac12}dx=+\infty$, $\int^\infty |a'/a|^\gamma a^{(\gamma-1)/2}dt<\infty$ for some γ on $[1, 2]$, and $\int^\infty |\omega/a|dt<\infty$, then (*) has linearly independent solutions y_i such that

$$y_1=a^{-\frac12}\exp[-\textstyle\int^x a^{\frac12}dt][1+o(1)],$$
$$y_2=a^{-\frac12}[1+o(x)]$$
$$y_3=a^{-\frac12}\exp.\ [\textstyle\int^x a^{\frac12}dt][1+o(1)].$$

N. D. Kazarinoff (Ann Arbor, Mich.)

2514:

Sibuya, Yasutaka. Sur réduction analytique d'un système d'équations différentielles ordinaires linéaires contenant un paramètre. J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1958), 527–540.

Let (1) $\varepsilon^\sigma dy/dx=A(x, \varepsilon)y$ be an n-dimensional vectorial linear differential equation, where σ is a positive integer and the matrix $A(x, \varepsilon)$ is holomorphic with respect to the complex variables x and ε in the domain $|x|<\delta_0$, $0<|\varepsilon|<\rho_0$, $|\arg\varepsilon|<\alpha_0$. Assume that $A(x, \varepsilon)$ admits in this region an asymptotic expansion $A(x, \varepsilon)\sim\sum_{\nu=0}^\infty A_\nu(x)\varepsilon^\nu$ with holomorphic coefficients, as $\varepsilon\to 0$. If the eigenvalues of $A(x, 0)$ are distinct for $|x|<\delta_0$ (and also in certain more general cases), the asymptotic theory of the differential equation (1) is well known [see, e.g., H. L. Turrittin, Contributions to the theory of nonlinear oscillations, vol. II, Princeton, 1952, pp. 81–116; MR **14**, 377].

However, the canonic form of $A(x, 0)$ may change in $|x|<\delta_0$, and then major difficulties, frequently called "turning point problems", arise. Assume that $A(0, 0)$ possesses $s\leqq n$ distinct eigenvalues so that its canonical form is a block-diagonal matrix of s blocks along the diagonal, each block having just one eigenvalue μ_j, $(j=1, \cdots, s)$, with $\mu_j\neq\mu_k$, $j\neq k$. The main theorem of this paper establishes the existence of a coordinate transformation of the form (2) $y=P(x, \varepsilon)z$ which reduces (1) to an equation (3) $\varepsilon^\sigma dz/dx=B(x, \varepsilon)z$ whose matrix $B(x, \varepsilon)$ is block diagonal, such that for each block $B_j(x, \varepsilon)$ $(j=1, 2, \cdots, s)$ the matrix $B_j(0, 0)$ has μ_j as its only eigenvalue. Thus, all turning point problems can be reduced to problems where all eigenvalues of the leading matrix coalesce at the turning point. To prove this theorem the author constructs a formal series $\sum_{\nu=0}^\infty P(x)\varepsilon^\nu$ whose substitution for $P(x, \varepsilon)$ in the transformation (2) reduces (1) formally to a differential equation (3) with the desired properties. The proof that the formal series $\sum_{\nu=0}^\infty P(x)\varepsilon^\nu$ is the asymptotic expansion of a function $P(x, \varepsilon)$ that actually yields the reduction of the differential equation is given by deriving a system of nonlinear differential

equations for the elements of the matrix $P(x, \varepsilon)$ and by showing that these differential equations admit solutions with asymptotic power series expansions.

W. Wasow (Madison, Wis.)

2515:

Zadiraka, K. V. Investigation of a system of nonlinear differential equations containing a small parameter. Ukrain. Mat. Ž. 10 (1958), no. 2, 121–127. (Russian. English summary)

Consider a system in an n-vector x and an m-vector z:

$$(1)\qquad \dot{x}=f\left(x, z, t, \frac{t}{\mu}\right),\ \mu\dot{z}=F(x, z, t)$$

$$x(t_0)=x_0,\ z(t_0)=z_0.$$

The solution is compared for small μ with that of

$$(2)\qquad \dot{\bar{x}}=f_0(\bar{x}, \bar{z}, t),\ \bar{x}(t_0)=x_0,\ \bar{z}=\varphi(\bar{x}, t),$$

where $\varphi(x, t)$ is a root of $F(x, z, t)=0$ and

$$(3)\qquad f_0(x, z, t)=\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(x, z, t, v)dv.$$

When in (1) there is no t/μ, the system is of the type investigated by Tychonov [Mat. Sb. N.S. **22(64)** (1948), 193–204; MR **9**, 588] and Gradstein [Dokl. Akad. Nauk. SSSR (N.S.) **66** (1949), 789–792; MR **10**, 709]. When z, t are lacking, one has a standard system of a type discussed by Bogoliubov-Mitropolskiĭ [Asymptotic methods in the theory of nonlinear oscillations, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955; MR **17**, 368; pp. 369–376].

Assume that (t_0, x_0, z_0) is in the closed region Ω: $0\leqq t\leqq L$, $x\in G$, $\|z-\varphi(x, t)\|\leqq\rho_0$. Also, that in Ω, f is of class C^1 in z and satisfies a Lipschitz condition in x with fixed constant λ, that F is of class C^2 in Ω, and finally that $F(x, z, t)=0$ has an isolated class C^1 solution. Let $A=(\partial F/\partial z)|_{z=\varphi}$. Theorem: If the solution $\bar{x}(t)$ of (2) together with its δ-neighborhood is in G, if in that set the limit in (3) exists, and if the characteristic roots of A are to the left of some vertical line $y=-\alpha$, $\alpha>0$, then the solution $x(t, \mu)$ of (1) approaches $\bar{x}(T)$ as $\mu\to0$. [Additional reference: Lyaščenko, ibid. **96** (1954), 237–239; MR **16**, 133].

S. Lefschetz (Princeton, N.J.)

2516:

Kuzmak, G. E. On the theory of non-autonomous quasilinear systems with many degrees of freedom. Ukrain. Mat. Ž. 10 (1958), no. 2, 128–146. (Russian. English summary)

The system dealt with is

$$\ddot{x}+Ax=\varepsilon f(x, v(t), \varepsilon)$$

where x, f are p-vectors, A is a constant matrix, v a known q-vector $(q=n-p)$ and the components of f are polynomials in those of x, v and in ε. The search is for approximate asymptotic solutions for ε small, valid for a long time period, and it is done by methods à la Krylov-Bogoliubov, but with altogether more complicated auxiliary variables than the ones, amplitude and phase, which they utilized.

S. Lefschetz (Princeton, N.J.)

2517:

Utz, W. R. Properties of solutions of certain second order nonlinear differential equations. Proc. Amer. Math. Soc. 8 (1957), 1024–1028.

The nontrivial solutions $x(t)$ $(x(t)\not\equiv0)$ of the linear differential equation $x''+dx'+ex=0$, where d and e are positive constants, are either oscillatory or monotonically approach zero as $t\to\infty$. Sufficient conditions are given that this be true of the nontrivial solutions of

$$x''+f(x, x')+g(x)=0.$$

It is assumed that $x(t)\equiv0$ is a solution and that f and g are differentiable. It is shown to be sufficient, for example, that $f(x, x')\geqq0$ for all (x, x'), $xg(x)>0$ for all $x\neq0$, and $\int_0^x g\to\infty$ as $x\to\infty$. By placing further restrictions on f and g it is shown that the amplitudes of all oscillatory solutions must decrease monotonically. By restricting f to be of the form $f(x, x')=h(x)x'$, sufficient conditions are given that allow negative damping $(h(x)<0)$. The results are simple consequences of the boundedness and uniqueness of the solutions.

J. P. LaSalle (Baltimore, Md.)

2518:

Reĭzin', L. È. Behavior of the integral curves of a system of differential equations near a singular point in a space of more than one dimension. Latvijas PSR Zinātņu Akad. Vēstis 1958, no. 3(128), 107–120. (Russian. Latvian summary)

The system considered is

$$(1)\qquad \dot{x}_i=X_i(x_1, \cdots, x_n)\ (i=1, 2, \cdots, n),$$

where the X_i are forms of degree n with the origin as unique common zero. After some generalities, description of special directions of approach, and assertion that their number is finite, in fact, among the common solutions of

$$(2)\qquad x_iX_j-x_jX_i=0\ (i, j=1, 2, \cdots, n),$$

the author projects on a sphere of constant radius centered at the origin and gives a differential system for the projected paths. The special directions correspond to some of the critical points of the projection.

After this the author turns his attention to the case $n=3$. The projection may then be reduced to

$$\dot{u}=U=X_1(u, v, 1)-uX_3(u, v, 1)$$
$$(3)\qquad \dot{v}=V=X_2(u, v, 1)-vX_3(u, v, 1)$$
$$u=x_1/x_3,\ v=x_2/x_3,\ x_3=1.$$

The indices of a point and closed curve (containing no critical point) are defined. The multiplicity $\mu(P)$ of a point P is its multiplicity as intersection of $U=V=0$. Let $I(P)$ be the index of P. The following properties are proved: (a) $I(P)-\mu(P)$ is even; (b) $\mu(P)\leqq[I(P)]^2$; (c) if the system has the maximum number m^2+m+1 of critical points, then they are all simple and $\frac{1}{2}(m^2+m)+1$ have index $+1$ and the rest index -1; (d) $I(P)\leqq m$; (e) if $m+1$ critical points are on the same line L, then the intervals of L between consecutive critical points are paths; (f) for $m=2$ there are at most two foci or centers.

In a special case the author also investigates the (rather complicated) 3-space behavior corresponding to a critical point of (3).

S. Lefschetz (Princeton, N.J.)

2519:

Moser, Jürgen. On the generalization of a theorem of A. Liapounoff. Comm. Pure Appl. Math. 11 (1958), 257–271.

Consider the Hamiltonian system

$$(1)\qquad \dot{x}_\nu=\partial H/\partial y_\nu,\ \dot{y}_\nu=-\partial H/\partial x_\nu\ (\nu=1, \cdots, n),$$

where $H=H(x, y)$, $x=(x_1, \cdots, x_n)$, $y=(y_1, \cdots, y_n)$, is real analytic in the neighborhood of the origin, $H(0)=0$. Assume the eigenvalues α_i, $-\alpha_i$ $(i=1, \cdots, n)$, of the Jacobian matrix of (1) at the origin to be distinct, α_1, α_2 independent over the reals and $\alpha_i\neq n_1\alpha_1+n_2\alpha_2$ for all integers n_1, n_2, with $i\geqq3$. Then it is proved that there

exists a 4-parameter family of solutions of (1) of the form

(2) $x_\nu=\varphi_\nu(\xi_1, \xi_2, \eta_1, \eta_2)$, $y_\nu=\psi_\nu(\xi_1, \xi_2, \eta_1, \eta_2)$

with $\xi_\mu=\mathring{\xi}\exp(ta_\mu(\mathring{\zeta}))$, $\eta_\mu=\mathring{\eta}_\mu\exp(-ta_\mu(\zeta))$ $(\mu=1, 2)$, and $a_1(\mathring{\zeta})=\alpha_1+\cdots$, $a_2(\mathring{\zeta})=\alpha_2+\cdots$ being convergent power series in $\zeta_\mu=\mathring{\xi}_\mu\mathring{\eta}_\mu$. The series (2) converge in a neighborhood of the origin; the rank of $\partial(\varphi_\nu, \psi_\nu)/\partial(\xi_\mu, \eta_\mu)$ is four. The solution (2) depends on the four complex parameters $\mathring{\xi}_\mu$, $\mathring{\eta}_\mu$, but if $\alpha_1, \alpha_2, -\alpha_1, -\alpha_2$ contain their complex conjugates then we can have real solutions depending on four real parameters. The above theorem generalizes one of A. M. Liapounoff [Problème général de la stabilité du mouvement, Ann. Math. Studies, no. 17, Princeton Univ. Press, 1947; MR 9, 34; p. 386]. Two applications are indicated, one to the plane 3-body problem, the other to the motion of a charged particle in a magnetic field (Stormer's problem). *M. M. Peixoto* (Baltimore, Md.)

2520:

Vinogradov, N. N. On asymptotic stability in the large of the trivial solution of a dynamical system of two differential equations. Minsk. Gos. Ped. Inst. A. M. Gor'k. Uč. Zap. 7 (1957), 145–158. (Russian)

This paper deals with the asymptotic stability in the large of the planar system

(1) $$\dot{x}=F(x, y),\quad \dot{y}=f(x, y),$$

where $F=0$, $f=0$ solely at the origin and satisfy the standard existence and unicity conditions throughout the plane. We first have the very general Theorem. Suppose that: (a) the origin is stable; (b) every path remains at finite distance as $t\to\infty$; (c) there are no closed paths. Then the origin is asymptotically stable in the large. This proposition, while very general, is not easily applicable, so further restrictions more manageable are imposed. The first is that F and f are of class C^1 throughout the plane. Next, it is assumed that $F=0$, $f=0$ consist of a certain number of branches through the origin, each with an equation $y=\varphi(x)$, φ of class C^1, with $\varphi'(x)\neq0$ for $x\neq0$, and of fixed sign in each quadrant. Essentially under these circumstances the author examines various easily ascertained combinations and shows that in each case there is asymptotic stability in the large. The method followed throughout is consideration of the behavior of the paths in the various sectors produced by $F=0$, $f=0$. [References: Erugin, Akad. Nauk SSSR. Prikl. Mat. Meh. 14 (1950), 459–512, 659–664;16 (1952), 620–628; MR 14, 376; Malkin, ibid. 16 (1952), 365–368; Krasovskiĭ, ibid, 16 (1952), 547–554; MR 14, 376]. *S. Lefschetz* (Princeton, N.J.)

2521:

Kudakova, P. V. On stability in a finite time interval. Akad. Nauk Kazah. SSR. Trudy Sektor. Mat. Meh. 1 (1958), 41–45. (Russian)

Let the n-vector system

(1) $$\dot{x}=\omega(t, x)$$

be such that in the region

$$H: t\in[T_1, T_2],\ \|x\|\leqq h\ (T_1\neq T_2,\ h>0)$$

there passes a single trajectory through each point and that $\omega(t, 0)=0$. Let $V(t, x)\geqq0$ in H (h small). Then for each t, as above, $V=A$ bounds a region $\Omega_A(t): V\leqq A$ in the x space.

Definition. If, whatever t_0, x_0 such that $V(t_0, x_0)\leqq A$, the trajectory $x(t)$ with those initial values satisfies

$$V(t, x)\leqq A \text{ for } t\in[t_0, T]\ (T_1\leqq t_0\leqq T\leqq T_2),$$

then the origin is said to be stable on $[t_0, T]$ relative to $\Omega_A(t)(=\text{rel}\,\Omega)$, otherwise it is unstable on $[t_0, T]$ rel Ω.

Theorem 1: If $\dot{V}(t_0, x_0)>0$, the origin is unstable rel Ω. Theorem 2: If $\dot{V}(t, x)<0$ for $t\in[t_0, T]$, then the origin is stable rel Ω. Theorem 3: A necessary and sufficient condition for stability rel Ω is: $\dot{V}_0\leqq0$, $\dot{V}<0$ for $t\in[t_0, T]$.

Similar theorems are given under the assumption that (1) is of the form

(2) $$\dot{x}=P(t)x+L(t, x),$$

where L is small relative to the first term, in comparison with the first approximation

(3) $$\dot{x}=P(t)x.$$

Finally, assuming $P(t)=P(t_0)+Q(t)$, $Q(t_0)=0$, the expected stability and instability theorems are given in terms of the real parts of the characteristic roots of $P(t_0)$. [References: Kamenkov, Akad. Nauk. SSSR. Prikl. Mat. Meh. 17 (1953), 529–540; MR 15, 795; A. A. Lebedev, ibid. 18 (1954), 139–148; MR 16, 132]. *S. Lefschetz* (Princeton, N.J.)

2522:

Harasahal, V. On stability of linear systems of linear differential equations of second order. Akad. Nauk Kazah. SSR. Trudy Sektor. Mat. Meh. 1 (1958), 46–49. (Russian)

In the system

(1) $$\dot{x}=p_{11}x+p_{12}y,\quad \dot{y}=p_{21}x+p_{22}y,$$

let $p_{ij}=p_{ij}(t)$ be differentiable for $t\geqq0$ and such that

(2) $$p_{11}+p_{22}<0,\quad p_{11}p_{22}-p_{12}p_{21}>0.$$

If the p_{ij} were constant, (2) would guarantee the asymptotic stability of the origin for all initial values of a motion. To obtain a similar result, supplementary conditions must be imposed on the $p_{ij}(t)$. They are derived as follows. Take $\Phi=Ax^2+2Bxy+Cy^2$. It is known that one may choose Φ uniquely as a positive definite form such that

$$\Phi_x(p_{11}x+p_{12}y)+\Phi_y(p_{21}x+p_{22}y)=-2(x^2+y^2).$$

Comparing coefficients, one obtains A, B, C from linear equations with determinant $\Delta>0$, in view of (2). Take $v=\Delta\Phi$. Then v is a quadratic form in x, y and, upon writing down that it is definite negative, one obtains the required necessary and sufficient condition. These are developed explicitly for the system (1) equivalent to

$$\ddot{x}+p(t)\dot{x}+q(t)x=0,$$

with p, q differentiable for $t\geqq0$. [Reference: Gorelik, Ž. Tehn. Fiz. 4, no. 10, (1934); 5, no. 2, 3, (1935).] *S. Lefschetz* (Princeton, N.J.)

2523:

Bedel'baev, A. K. Some criteria for distinguishing dangerous and harmless sections of the boundary of a region of stability of a class of self-regulating systems. Akad. Nauk Kazah. SSR. Trudy Sektor. Mat. Meh. 1 (1958), 50–61. (Russian)

In a certain case in which the characteristic roots of the coefficient matrix of the first degree terms included a pair of pure conjugate complex roots, Malkin [Theory of stability of motion, Gosudarstv. Izdat. Tehn. Teor. Lit. Moscow-Leningrad, 1952; MR 15, 873; p. 164] dealt with stability by means of a highly ingenious but com-

plicated transformation. This is applied here in two situations where: (a) there are two pairs of pure conjugate complex roots; (b) there is one such pair and a zero root.

S. Lefschetz (Princeton, N.J.)

2524:

Harasahal, V. On the characteristic numbers of linear systems of differential equations with variable coefficients. Akad. Nauk Kazah. SSR. Trudy Sektor. Mat. Meh. 1 (1958), 147–150. (Russian)

In considering the stability problem for linear systems of the form $X'=PX$, where $X(t)$ is an $n\times 1$ matrix and $P(t)$ an $n\times n$ matrix, continuous and uniformly bounded for $t\geqq 0$, the author considers the possibility that X^TX is a Liapunov function [S. Lefschetz, Differential equations: geometric theory, Interscience, New York, 1957; MR 20 #1005; pp. 112, 134], and obtains both a positive and a negative theorem. Let $\alpha(t)$ and $\beta(t)$ be the smallest and largest eigenvalues of the matrix $Q=(P+P^T)/2$; $a=-\limsup t^{-1}\int_{t_0}^t \beta(\tau)d\tau$; $b=-\limsup t^{-1}\int_{t_0}^t \alpha(\tau)d\tau$. Theorem: The characteristic numbers of $X'=PX$ lie in the interval $[a, b]$. If $a<0$, the solution $X=0$ is asymptotically stable. If $b>0$, $X=0$ is unstable.

R. R. Kemp (Kingston, Ont.)

2525:

Štelik, V. G. On the determination of a finite time interval of the stability of solutions of a system of differential equations. Ukrain. Mat. Ž. 10 (1958), no. 1, 100–102. (Russian)

Consider an n-vector system

$$\dot{x}=P(t)x+X(t, x),$$

where the components of X are power series in those of x, beginning with terms of degree $\geqq 2$, and the terms of the matrix P satisfy a Lipschitz condition on $[t_0, T]$, $t_0\geqq 0$. The X series converge in $\|x\|<h$ and t as above. Kamenkov [Akad. Nauk SSSR. Prikl. Mat. Meh. 17 (1953), 529–540; MR 15, 795] has given this definition of stability in $[t_0, t_0+\tau]$: Given any small ε and arbitrary nonsingular matrix A, if $\|A\cdot x(t_0)\|<\varepsilon$ implies $\|A\cdot x(t)\|<\varepsilon$ for $t\in[t_0, t_0+\tau]$, then the origin is stable on the interval of length τ. If this can only happen for $\tau=0$, then we have instability. Kamenkov [loc. cit.] and A. A. Lebedev [ibid. 18 (1954), 75–94; MR 16, 132] have given estimates for τ. Using a transformation due to Petrovskii [Byull. Moskov. Gos. Univ. Mat. 1 (1938), no. 7], the author gives an explicit procedure for calculating τ.

S. Lefschetz (Princeton, N.J.)

2526:

Wasow, Wolfgang. Solution of certain nonlinear differential equations by series of exponential functions. Illinois J. Math. 2 (1958), 254–260.

Consider the differential system

$$(1)\qquad y'=f(y)+\sum_{k=1}^{m} g_k e^{i\omega_k x},$$

where $y=(y_1, \cdots, y_n)$ and g_k are vectors, the ω_k are real and dash indicates differentiation with respect to the independent variable x. If $v=(v_1, \cdots, v_n)$, call $\|v\|=\max_j |v_j|$, $j=1, \cdots, n$. Assume that: (i) the equation $f(y)=0$ has a solution $y=a_0$; (ii) the components of the vector $f(y)$ are holomorphic functions of $y_1, \cdots, y_n$ in the domain $\|y-a_0\|\leqq\rho$, ρ being a positive constant; (iii) the jacobian matrix $\partial f/\partial y$ at the point $y=a_0$ has no purely imaginary eigenvalue. It is then proved that, when the g_k are small enough, there is a particular solution of (1) of the form $y=a_0+\sum_{r=1}^\infty a_r e^{i\mu_r x}$, converging absolutely and uniformly for $-\infty<x<\infty$; the μ_r are of the type $\sum_{k=1}^m n_k\omega_k$, the n_k being non-negative integers not all zero, and the vectors a_r, $r>1$, can be recursively calculated by solving systems of linear equations. A similar perturbation theorem with a more general perturbed term but with a linear $f(y)$ was established by G. I. Biryuk [Dokl. Akad. Nauk. SSSR (N.S.) 96 (1954), 5–7; MR 16, 130).

M. M. Peixoto (Baltimore, Md.)

2527:

Klokov, Yu. A. Some theorems on boundedness of solutions of ordinary differential equations. Uspehi Mat. Nauk (N.S.) 13 (1958), no. 2(80), 189–194. (Russian)

Consider the equation

$$(1)\qquad \ddot{x}+a(t)f(x)=0.$$

It is assumed that a is of class C^2 for $t\geqq 0$ and that $f(x)$ is continuous for every x and satisfies the usual conditions for the existence and unicity of solutions.

Theorem 1. Let $a(t)$ be $\geqq 0$ and monotone increasing. Let also $F(x)=\int_0^x f(x')dx'\to+\infty$ with $|x|$. Then all the solutions of (1) are bounded.

A solution of (1) is said to be regular when it is definite and of class C^2 for $0\leqq t_0\leqq t$. Theorem 2. All the solutions of $\ddot{x}+(1+\varphi(t))f(x)=0$ are bounded if $F(x)$ behaves as in Th. 1 and $\int_0^\infty |\varphi'(t)|dt<\infty$, $\varphi(t)\to 0$ as $t\to+\infty$.

Let $x=(x_1, \cdots, x_n)$ with Euclidean norm. Theorem 3. Let $F(x)$ be of class C^2 and let $\min_{\|x\|\leqq r} F(x)\to+\infty$ with $r\to+\infty$. Let $a_i(t)$ be positive and monotone increasing functions. Then all the solutions of

$$\ddot{x}_i+a_i(t)\frac{\partial F}{\partial x_i}=0$$

are bounded. Theorem 4. Consider

$$\ddot{x}_i+(1+\varphi_i(t))\partial F/\partial x_i=0.$$

If the φ_i behave like φ in Th. 2 and F as in Th. 3, then all the regular solutions of the system are bounded.

S. Lefschetz (Princeton, N.J.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 2563a-b, 2640, 2960, 2961, 2988.

2528:

Birkhoff, Garrett; and Mullikin, Thomas. Regular partial differential equations. Proc. Amer. Math. Soc. 9 (1958), 18–25.

Let $D_i=\partial/\partial x_i$ denote a partial differential operator, and $\|p_{jk}(\mathbf{D})\|$ denote a general matrix of polynomials with constant coefficients in the differential operators D_i. The authors consider the Cauchy problem for the system of equations

$$\frac{\partial u_j}{\partial t}=\sum p_{jk}(D_1, \cdots, D_r)u_k(x_1, \cdots, x_r; t)\ (j, k=1, \cdots, n),$$

subject to the condition $\mathbf{u}(\mathbf{x}; 0)=\mathbf{v}_0(\mathbf{x})$, where $\mathbf{v}_0(\mathbf{x})$ is supposed given in infinite $(x_1, \cdots, x_r)$-space. If the system is "regular", then they interpret the symbolic solutions of the system, globally, as the semi-orbits, for $t>0$, of a C_0-semi-group acting on an appropriate Banach space. Conditions are also given under which these solutions become literal solutions of the system.

K. Chandrasekharan (Bombay)

2529:

Èĭdel'man, S. D. Some theorems on the stability of

the solutions of parabolic systems. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 253–255. (Russian)

This paper presents some theorems (without proof) on the stability (in the sense of Liapounov) of the solution of the parabolic equation

$$(1)\qquad \frac{\partial u}{\partial t}=\sum_{k=0}^{2b} P_k\left(t,\frac{1}{i}\frac{\partial}{\partial x}\right)u \quad (t\geqq 0),$$

where P_k is a differential operator of order k with continuous coefficients. The results on the question of the stability of the fundamental solution of the system (1) are obtained through the study of the Green's matrix associated with (1). *C. G. Maple* (Ames, Iowa)

2530:

Titchmarsh, E. C. On the eigenvalues in problems with spherical symmetry. Proc. Roy. Soc. London. Ser. A. **245** (1958), 147–155.

The problem discussed is that of the distribution of the eigenvalues of

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\{\lambda-q(r)\}\psi=0,$$

where $r^2=x^2+y^2+z^2$. It is assumed that $q(r)$ is a three-times differentiable function of r which tends to infinity with r, that $q(r)$ and $q'(r)$ are non-decreasing functions of r, that $q(0)=0$ and that, as $r\to\infty$, $q'(r)/q(r)$, $q''(r)/q'(r)$ and $q'''(r)/q''(r)$ are all of the form $O(1/r)$. The required eigenvalues are then the roots of equations of the form

$$\int_0^R \{\lambda-q(r)\}^{\frac{1}{2}}dr=(\tfrac{1}{2}l+n+\tfrac{3}{4}\pi)+\delta_n,$$

where l and n are non-negative integers, R is the root of the equation $q(r)=\lambda$, and δ_n tends to zero as $n\to\infty$. *E. T. Copson* (St. Andrews)

2531:

Cinquini Cibrario, Maria; e Cinquini, Silvio. Sopra una nuova estensione di un teorema di esistenza per equazioni a derivate parziali del primo ordine. Ann. Mat. Pura Appl. (4) **43** (1957), 51–81.

The question is of the existence of a solution of the Cauchy problem for the first order partial differential equation

$$(*)\qquad \frac{\partial z}{\partial x}=f(x,y_1,\cdots,y_n,z,q_1,\cdots,q_n),$$

where $z=z(x,y_1,\cdots,y_n)$ and $q_i=\partial z/\partial y_i$, $i=1,\cdots,n$. The Cauchy datum is $z(0,y_1,\cdots,y_n)=\phi(y_1,\cdots,y_n)$. Let f be defined in the "slab" $0\leqq x\leqq a_0$ and possess first partial derivatives with respect to each of the $2n+1$ variables y_i, z, q_i for each x. Then under rather weak continuity and Lipschitzian conditions on these functions and on the Cauchy datum, it is shown that there exists an a, $0<a\leqq a_0$, such that the "weak" problem

$$z(x,y_i)=\phi(y_i)+\int_0^x f(u,y_i,z(u,y_i),q_i(u,y_i))du$$

possesses, for $0\leqq x\leqq a$, a solution which is absolutely continuous in x for each n-tuple $(y_1,\cdots,y_n)$ and has continuous partial derivatives $\partial z/\partial y_i$ (in the reduced slab) which satisfy certain uniform Lipschitz conditions. Hence (*) is satisfied for almost all x in the interval $0\leqq x\leqq a$. Uniqueness follows from previous work of the second author. *C. R. DePrima* (Pasadena, Calif.)

2532:

Walsh, J. L.; and Young, David. Lipschitz conditions for harmonic and discrete harmonic functions. J. Math. Phys. **36** (1957), 138–150.

Given a function $u(x,y)$ harmonic in a square (or rectangle) R, whose boundary values are continuous and satisfy a Lipschitz condition of order α $(0<\alpha<1)$:

$$|u(x_1,y_1)-u(x,y)|\leqq L\delta^\alpha,\ \delta^2=(x-x_1)^2+(y-y_1)^2.$$

The authors show that u satisfies a Lipschitz condition of order α throughout R. They prove also the analogous theorem for discrete harmonic functions (solutions of an approximate difference equation, defined at the vertices of a square grid) with uniformity of the Lipschitz condition for all grid sizes h. Finally they apply their results to appraising degree of convergence as $h\to 0$ of the solution of the discrete Dirichlet problem to the solution of the continuous Dirichlet problem. Along the way, the authors prove some interesting theorems for u harmonic or discrete-harmonic in a half-plane or semi-infinite strip, with continuous Lipschitzian boundary values. *M. A. Hyman* (Yorktown Heights, N.Y.)

2533:

Simoda, Seturo. Notes pour la théorie des équations aux dérivées partielles du type elliptique. Mem. Osaka Univ. Lib. Arts Ed. Ser. B. no. **5** (1956), 5–15.

In the first part of the paper the author lists a number of postulates for an operator A (these are all satisfied by second order linear elliptic operators), and derives a number of rather immediate consequences. However, the most important result of the paper, occurring in the last part, may be roughly stated as follows. Let

$$\Phi_{(u)}(x,\lambda)\equiv\Phi(x_i,u,\partial u/\partial x_i,\partial^2u/\partial x_i\partial x_j,\lambda)$$

be a second order non-linear partial differential operator, elliptic for u belonging to a certain set S of functions defined in a domain d in E^n, and depending on a parameter $\lambda\in\Lambda$ (≡interval, say). Suppose (1) $\Phi_{(v)}(x,\lambda)-\Phi_{(w)}(x,\lambda)\geqq 0$ for a particular pair v, w in S, and for all (x,λ) in $d\times\Lambda$; (2) $w(x,\lambda)-v(x,\lambda)>0$ for some λ and all x in d; and (3) $\liminf\{w(x_n,\lambda_n)-v(x_n,\lambda_n)\}>0$ for all convergent $\{\lambda_n\}$ and $\{x_n\}$ having no limit point in d. Then (2) holds for all x, λ in $d\times\Lambda$. The crux of the proof is to write the difference (1) as a linear operator L acting on the difference (2), using the connectedness of Λ and applying Hopf's maximum principle to L. [See also S. Simoda and M. Nagumo, Proc. Japan Acad. **27** (1951), 334–339; and H. Westphal, Math. Z. **51** (1949), 690–695; MR **13**, 656; **11**, 252.] *W. Littman* (Berkeley, Calif.)

2534:

Čerpakov, P. V. On limit relations between solutions of equations of parabolic and of elliptic type. Kuĭbyšev. Aviacion. Inst. Trudy **2** (1954), 3–7. (Russian)

In this note the author uses the fundamental solutions of the heat equation and of the Laplace equation to indicate proofs of the limiting relations between transient and steady-state temperature distributions. *R. R. Kemp* (Kingston, Ont.)

2535:

Fox, David William. Sur le principe de Huygens pour un problème singulier de Cauchy. C. R. Acad. Sci. Paris **246** (1958), 213–215.

Verfasser behandelt die hyperbolische Differentialgleichung

$$\frac{\partial^2 u}{\partial t^2}+\frac{2-k}{t}\frac{\partial u}{\partial t}=\sum_{i=1}^{m}\left(\frac{\partial^2 u}{\partial x_i^2}+\frac{\lambda_i}{x_i^2}u\right)$$

mit den singulären Cauchyschen Anfangsbedingungen

$u=f(x)$, $u_t=0$ für $t=0$ und $f(x)=f(x_1, \cdots, x_m)$. Die Lösung dieses singulären Cauchyschen Problems lautet

$$u_\lambda^{(k)}(x, t, f)=\frac{\Gamma(\frac{1}{2}(k+1))}{\pi^{m/2}\Gamma(\frac{1}{2}(k-m+1))} t^{1-k} \int f(\xi)\gamma_\lambda^{(2-k)}(x, \xi, t)d\xi.$$

Dabei gilt $\Gamma=t^2-\sum_i (x_i-\xi_i)^2$ und $\sum_i (x_i-\xi_i)^2 \leqq t^2$. Die Lösungsformel gilt für $k>m-1$ und kann durch Reihenentwicklung der rechten Seite auch noch für $k\leqq m-1$, $k\neq -1, -3, \cdots$ analytisch fortgesetzt werden. Für $k\neq -1, -3, \cdots$ gilt das Huygensche Prinzip genau dann, wenn $m-k=2n+1$, $n=0, 1, 2, \cdots$, $\lambda_i=-\gamma_i(\gamma_i+1)$, $\gamma_i=0, 1, 2, \cdots$, $\sum_i \gamma_i \leqq n$. Die Funktionen $\gamma_\lambda^{(k)} \geqq t^{1-k}\gamma_\lambda^{(2-k)}$ sind Lösungen der Differentialgleichung.

M. Pinl (Köln)

2536:

Blondel, Jean-Marie. Sur le comportement des solutions d'une équation linéaire hyperbolique du second ordre, au voisinage de la singularité d'un coefficient. C. R. Acad. Sci. Paris **246** (1958), 36–38.

Etude du comportement des solutions de l'équation

$$(x-y)^n \frac{\partial^2 z}{\partial x \partial y}=A(x, y)z,$$

$$z(x, x-h)=f(x), \; h>0; \; \left(\frac{\partial z}{\partial x}\right)(x, x-h)=g(x),$$

$A(x, y)$ donnée une fois continûment différentiable, lorsque $x-y\to+0$. Trois cas sont à considérer selon que $0<n<1$; $1\leqq n<2$; $2\leqq n$; on montre que les résultats obtenus sont les meilleurs possibles, à l'aide du cas $A=$ constante.

Problème analogue pour $x^n(\partial^2 z/\partial x\partial y)=A(x, y)z$, $z(x, 0)=f(x)$, $z(h, y)=g(y)$, $x\to+0$.

J. L. Lions (Nancy)

2537:

Hellwig, Günter. Über die Anwendung der Laplace-Transformation auf Ausgleichsprobleme. Math. Nachr. **18** (1958), 281–291.

An existence theorem is established for a solution of the boundary value problem

$$b(x)u_t=[p(x)u_x]_x-q(x)u \; (l<x<m, \, t>0), \; u(x, 0)=0,$$
$$a_{i1}u(l, t)+a_{i2}u_x(l, t)+a_{i3}u_t(l, t)+b_{i1}u(m, t)$$
$$+b_{i2}u_x(m, t)+b_{i3}u_t(m, t)=f_i(t) \quad (i=1, 2),$$

where the prescribed functions b, p, q and f_i satisfy certain conditions of continuity and the constant coefficients a_{ij} and b_{ij} in the boundary conditions are restricted in a natural manner. The author represents his solution $u(x, t)$ by an inversion integral for the Laplace transformation. The integrand is $e^{st}v(x, s)$, where v is the solution of the problem in ordinary differential equations that arises when the Laplace transformation is applied formally to the problem in $u(x, t)$. The boundary conditions are divided into three classes, in order to simplify the analysis. His results are found by establishing order properties of $v(x, s)$ in the complex plane of s. Several properties of continuity of the solution $u(x, t)$ and its derivatives are also determined. The results and methods are extensions of those published earlier by the author [Math. Z. **66** (1957), 371–378; MR **18**, 903] and by the reviewer [Amer. J. Math. **61** (1939), 651–664; MR **1**, 57].

R. V. Churchill (Ann Arbor, Mich.)

2538:

Mitrinovitch, Dragoslav S. Sur certaines équations aux dérivées partielles à deux fonctions inconnues. Bull. Soc. Math. Phys. Serbie **8** (1956), 3–6. (Serbo-Croatian summary)

The author considers several indeterminate partial differential equations such as

$$\frac{1}{u}\,\frac{\partial^2 u}{\partial x\partial y}=\frac{k}{v}\,\frac{\partial^2 v}{\partial x\partial y} \; (k=\text{const.}),$$

involving two unknown functions of two variables. The equations are solved by setting $u=T(v)$, where T is an arbitrary, sufficiently differentiable function. The transformed equation can then be solved for v, giving rise to solutions $\{u, v\}=\{T(v), v\}$ of the original equation. A similar method is also applied to one equation which is not of the above type, namely,

$$\frac{\partial^2 f}{\partial t^2}+\frac{a'}{a}\left(\frac{\partial^2 f}{\partial y^2}+k\frac{\partial f}{\partial y}\right)+\left(\frac{a'}{a}-\frac{a''}{a'}\right)\frac{\partial f}{\partial t}=0,$$

where a is a function of t and k is a constant.

J. Elliott (New York, N.Y.)

2539:

Višik, M. I.; and Lyusternik, L. A. Regular degeneration and boundary layer for linear differential equations with small parameter. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 5(77), 3–122. (Russian)

On considère des opérateurs différentiels $L_\varepsilon=L_0+L_{1,\varepsilon}$ où $L_{1,\varepsilon}$ est un opérateur différentiel d'ordre plus élevé que L_0, et à coefficients dépendant de ε, tendant vers 0 lorsque $\varepsilon\to 0$. Tous les opérateurs considérés sont linéaires. On considère des problèmes aux limites A_0 [resp. A_ε] attachés à L_0 [resp. L_ε]. Soit u_0 [resp. u_ε] solution de $A_0u_0=h$ [resp. $A_\varepsilon u_\varepsilon=h$], u_0 et u_ε vérifiant les conditions aux limites correspondantes. Le problème est de voir si, lorsque $\varepsilon\to 0$, $u_\varepsilon\to u_0$, à quel sens, et d'évaluer, s'il y a lieu, la différence $u_\varepsilon-u_0$ (dans des normes variées). Ce travail apporte de nombreux et intéressants compléments aux problèmes analogues déjà étudiés par de nombreux auteurs [pour un état de la question, antérieurement à cet article, voir K. O. Friedrichs, Bull. Amer. Math. Soc. **61** (1955), 485–504; MR **17**, 615].

Le plan est le suivant; dans les § 1, 2, 3, on considère le cas des équations diffèrentielles, à coefficients constants (§ 1), puis variables assez réguliers (§ 2, 3). Dans les § 4, $\cdots$, 9, on considère des opérateurs différentiels elliptiques, les conditions aux limites étant celles de Dirichlet. De façon plus précise, on considère au § 4 l'opérateur

$$L_\varepsilon=\varepsilon L_2+L_1, \text{ dans } R^2 \text{ (variables } x, y),$$

$$L_1=\frac{\partial}{\partial x}-f(x, y),$$

$$L_2=a(x, y)\frac{\partial^2}{\partial x^2}+2b(x, y)\frac{\partial^2}{\partial x\partial y}+c(x, y)\frac{\partial^2}{\partial y^2}$$
$$+d(x, y)\frac{\partial}{\partial x}+e(x, y)\frac{\partial}{\partial y}+g(x, y).$$

Le problème A_ε est le problème de Dirichlet, dans un ouvert Q de frontière Γ; dans le problème A_0 on demande à u_0 de s'annuler sur une partie convenable Γ_+ de la frontière. Dans les § 6, 7, 8, on considère le cas d'opérateurs uniformément elliptiques, d'ordres quelconques, à un nombre quelconque de variables.

Le § 9 étudie, dans le cas elliptique, avec données de Dirichlet dans un ouvert borné, le comportement des valeurs propres et des fonctions propres lorsque $\varepsilon\to 0$. Le § 10 étudie enfin le cas d'équations paraboliques.

Voici un résultat typique, correspondant au cas le plus simple (§ 1); L_0 est un opérateur différentiel sur R, d'ordre k, à coefficients constants et à racines caractéris-

tiques simples: $\mu_1, \cdots, \mu_k$. Soit

$$L_\varepsilon = L_0 + \sum_{r=1}^{l} \varepsilon^r a_{k+r} D^{k+r} \quad (D = d/dx).$$

On désigne par A_0 [resp. A_ε] le problème: trouver y_0 [resp. y_ε] solution de $L_0 y_0 = 0$ [resp. $L_\varepsilon y_\varepsilon = 0$] avec $D^i y_0(0) = D_i$, $D^j y_0(1) = E_j$, $i=0, \cdots, k_1-1$; $j=0, \cdots, k_2-1$; $k_1+k_2=k$ [resp. $D^i y_\varepsilon(0)=D_i$, $D^j y_\varepsilon(1)=E_j$, $i=0, \cdots, l_1-1$; $j=0, \cdots, l_2-1$; $l_1 \geq k_1$; $l_2 \geq k_2$; $l_1+l_2=l$]. Soit $Q_0(\lambda) = \sum_{r=0}^{l} a_{k+r}\lambda^r$. On dit que A_ε dégénère régulièrement dans A_0 si $Q_0(\lambda)=0$ admet l_1 [resp. l_2] racines de partie réelle négative, soit $-\lambda_1, \cdots, -\lambda_{l_1}$, [resp. positive, soit $\nu_1, \cdots, \nu_{l_2}$].

Les racines de $P_\varepsilon(\lambda)=0$ peuvent se représenter sous la forme $\bar{\mu}_i = \mu_i + \varepsilon_i$, $i=1, \cdots, k$; $-\bar{\lambda}_i/\varepsilon = -(\lambda_i + \varepsilon'_i)/\varepsilon$, $i=1, \cdots, l_1$; $\bar{\nu}_i/\varepsilon = (\nu_i + \varepsilon_i'')/\varepsilon$, $i=1, \cdots, l_2$; où ε_i, ε_i', $\varepsilon_i'' \to 0$ lorsque $\varepsilon \to 0$. Dans ces conditions, si A_ε dégénère régulièrement en A_0, $y_\varepsilon - y_0 = v_\varepsilon + z_\varepsilon$ où v_ε=combinaison linéaire à coefficients constants des l_1 termes $\varepsilon^{k_1}\exp(-\bar{\lambda}_i x/\varepsilon)$ et des l_2 termes $\varepsilon^{k_2}\exp(\bar{\nu}_i(x-1)/\varepsilon)$, et où $z_\varepsilon \to 0$ uniformément sur [0, 1] ainsi que chacune de ses dérivées.

Voici une idée de la méthode du § 2. On prend $L_0 = \sum_0^k a_j(x) D^j$, $L_\varepsilon = L_0 + \sum_{r=1}^{l} \varepsilon^r a_{k+r}(x) D^{k+r}$; soit w_0 avec $L_0 w_0 = h$, $D^i w_0(0)=0$, $i=0, \cdots, k-1$; u_ε avec $L_\varepsilon u_\varepsilon = h$, $D^i u_\varepsilon(0) = 0$, $i=0, \cdots, k+l-1$. On considère un développement limité de $a_j(x)$ à l'origine; pour simplifier

(1) $\quad a_j(x) = a_{j,0} + a_{j,1,0}(x)x$ ($N=0$ dans les notations des A.).

Soit $Q_0(\lambda) = \sum_{r=0}^{l} a_{k+r,0}\lambda^r$. On suppose que $Q_0(\lambda)=0$ admet l racines de partie réelle négative (dégénérescence régulière). Utilisant (1) et posant $x=\varepsilon t$, $\varepsilon^k L_\varepsilon = M_0 + \varepsilon R_1$, $M_0 = \sum_{r=0}^{l} a_{k+r,0} D^{k+r}$, $D=d/dt$, R_1 à coefficients variables. Soit $M_0 = D^k \tilde{M}_0$, v_0 solution de $\tilde{M}_0 v_0 = 0$, $D^{k+r}(w_0+v_0)(0) = 0$, $r=0, \cdots, l-1$. Grâce à l'hypothèse de dégénérescence régulière, $v_0 = \sum_{i=1}^{l} c_i \exp(-\lambda_i x/\varepsilon)$, Re $\lambda_i > 0$. Si l'on pose $\alpha_0 = -\sum_{i=1}^{l} c_i \sum_{s=0}^{k-1} (-\lambda_i t)^s/s!$, on a: $X = w_0 + v_0 + \varepsilon\alpha_0$ vérifie $D^i X(0) = 0$, $i=0, \cdots, k+l-1$; et si $z_0 = u_\varepsilon - (w_0 + v_0 + \varepsilon\alpha_0)$, on a: $L_\varepsilon z_0 = -\varepsilon g_0$, $D^i z_0(0) = 0$, pour $i=0, \cdots, k+l-1$, g_0 étant bornée. De la sorte z_0 est "petit" et on a une première évaluation de u_ε.

Pour le cas des opérateurs différentiels en dimension >1, on prend de nouveaux systèmes de coordonnées, $\rho, \varphi_1, \cdots, \varphi_{n-1}$, ρ désignant la distance à la frontière. On effectue des développements limités des coefficients en ρ (en considérant les φ_i comme fixés) et on suit une méthode analogue à celle esquissée ci-dessus.

J. L. Lions (Nancy)

2540:

Huet, Denise. Phénomènes de perturbation singulière. C. R. Acad. Sci. Paris **246** (1958), 2096–2098.

Suite d'une note antérieure [C. R. Acad. Sci. Paris **244** (1957), 1438–1440; MR **19**, 421]. Soit u_ε la solution d'un problème aux limites elliptique: $(\varepsilon A + B)u = f$; l'A. a montré que sous certaines conditions, $u_\varepsilon \to u$ lorsque $\varepsilon(>0) \to 0$, u étant la solution de $Bu=f$, avec des conditions aux limites convenables. La convergence a lieu dans l'espace de Hilbert, domaine de $B^{\frac{1}{2}}$. On suppose dans cette note que les coefficients de A et B et la frontière de l'ouvert Ω borné dans lequel on résout les problèmes aux limites sont suffisamment réguliers. On montre alors que la convergence a lieu dans les espaces du type $H^k(\Omega)$, k arbitrairement grand ($H^k(\Omega)$ désigne l'espace de Hilbert des fonctions $\in L^2(\Omega)$ ainsi que toutes leurs dérivées jusqu'à l'ordre k). Les démonstrations utilisent entre autres les résultats et méthodes de L. Nirenberg [Comm. Pure Appl. Math. **8** (1955), 649–675; MR **17**, 742]. Les résultats de l'A. sont moins précis que ceux de Visik et Lousternik [voir #2539 ci-dessus], mais valables pour des conditions aux limites beaucoup plus variées.

J. L. Lions (Nancy)

2541:

Saul'ev, V. K. On a class of elliptic equations solvable by the method of finite differences. Vyčisl. Mat. **1** (1957), 81–86. (Russian)

Let Q be a domain in Euclidean n-space whose boundary Γ is m times continuously differentiable. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$. The functions $a_{\alpha\beta} = a_{\beta\alpha}$, b_α ($|\alpha|=|\beta|=m$), and c are assumed continuous in Q. The author considers the problem of finding a (weak) solution to the equation

$$\sum_{|\alpha|=|\beta|=m} D^\alpha(a_{\alpha\beta}D^\beta u) + \sum_{|\alpha|=m} b_\alpha D^\alpha u + cu = \varphi,$$

satisfying the condition $D^\gamma u = 0$ on Γ, $|\gamma| = 0, 1, \cdots, m-1$. The condition

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\xi_\alpha\xi_\beta \geq \nu \sum_{|\alpha|=m} \xi_\alpha^2$$

(ν=const>0) of ellipticity is assumed. {The author's cumbersome notation has been simplified.} The method consists of replacing the differential equation by a system of difference equations with mesh h. It is shown (Theorem 1) that, under a certain additional condition, this system of difference equations has a solution $u^{(h)}$ for arbitrary φ, and (Theorem 2) that, for $h \to 0$, the point solution $u^{(h)}$, if extended polylinearly over Q, converges in the mean to a weak solution (whose existence is not assumed) of the boundary value problem. This generalizes previous results of S. L. Sobolev and O. A. Ladyzhenskaja. In the proof of Theorem 1 a finite difference analog of an inequality ascribed to Friedrichs plays an important part.

P. Henrici (Los Angeles, Calif.)

2542:

Roşculeţ, Marcel N. Sur une classe d'équations aux dérivées partielles. Bul. Inst. Politehn. Bucureşti **18** (1956), no. 1–2, 5–10. (Romanian. Russian and French summaries)

The author extends the method of Almansi [Ann. Mat. Ser. III **2** (1899), 1–51] to find solutions of a generalized Laplacian.

J. A. Ward (Holloman A.F.B., N.M.)

2543:

Mihaĭlov, V. P. Non-analytical solutions of Goursat's problem for a system of differential equations in two independent variables. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 759–762. (Russian)

The author announces three theorems setting forth sets of sufficient conditions under which the following homogeneous Goursat problem is correctly posed: to find in the entire xt-plane (or in some neighborhood of the origin) a solution to the system $\partial u_i/\partial t = \sum_{j=1}^{n} b_{ij}(\partial u_j/\partial x)$ subject to the conditions $u_i(l_j) = \phi_i(t)$ ($i=1, \cdots, n$), in which l_i is the line $x = \mu_i t$ ($-\infty < t < \infty$); b_{ij}, μ_j are constant; $\phi_i(t)$ is everywhere continuously differentiable and does not increase faster than some power of $|t|$ as $|t| \to \infty$.

R. N. Goss (San Diego, Calif.)

2544:

Roşculeţ, Marcel N. Relations intégrales caractérisant les solutions de certaines équations aux dérivées partielles d'ordre fini ou infini. Acad. R. P. Romîne. Stud. Cerc. Mat. **8** (1957), 131–161. (Romanian. Russian and French summaries)

"Dans l'introduction, l'auteur donne des indications bibliographiques sur les principaux résultats obtenus par W.

Blaschke, S. Saks, N. Ciorănescu, W. J. Trjitzinsky, point de départ de son travail.

Dans la première partie, il indique que si $Q_s(\xi_1, \xi_2, \cdots, \xi_p)$ est un polynôme arbitraire de degré s des coordonnées $(\xi_1, \xi_2, \cdots, \xi_p)$ d'un point de la sphère Σ_p, de rayon égal à l'unité, de l'espace à p dimensions, on peut disposer du degré s, ainsi que des paramètres qui interviennent dans Q_s de sorte que

$$\lim_{r\to 0} \frac{1}{r^n}\int_{\Sigma_p} Q_s(\xi_1, \cdots, \xi_p) f(x_1+r\xi_1, \cdots, x_p+r\xi_p)\, d\delta_p = E_n(f),$$

où $f(x_1, \cdots, x_p)$ est une fonction continue, de même que ses dérivées, jusqu'à l'ordre n inclusivement, définie dans un domaine (D) qui contient la sphère de centre $(x_1, \cdots, x_p)$ et le rayon r, $E_n(f)$ étant une équation aux dérivées partielles, homogène, d'ordre n, dont les coefficients dépendent des coefficients de Q_s. Si $E_n(f)=0$ est une équation aux dérivées partielles d'ordre n, à coefficients constants, homogène et arbitraire, en $x_1, \cdots, x_p$ on peut toujours déterminer un polynôme $Q_s(\xi_1, \cdots, \xi_p)$ tel que

$$\lim_{r\to 0} \frac{1}{r}\int_{\Sigma_p} Q_s(\xi_1, \cdots, \xi_p) \frac{\partial^{n-1} f(x_1+r\xi_1, \cdots, x_p+r\xi_p)}{\partial r^{n-1}}\, d\sigma_p = E_n(f).$$

Dans le chapitre 6, passant du domaine réel au domaine complexe, l'auteur montre que si $E_n(f)=0$ est une équation aux dérivées partielles, non-homogène, à coefficients constants d'ordre n des variables $x_1, x_2, \cdots, x_s$,

$$\text{(a)} \qquad E_n(f) = \sum_{p=0}^{n} \alpha_{m_1,\cdots,m_s} \frac{\partial^p f}{\partial x_1^{m_1} \partial x_2^{m_2} \cdots \partial x_s^{m_s}} = 0,$$

on peut toujours lui associer un polynôme de Fourier

$$S_n(\theta_1, \theta_2, \cdots, \theta_s, r) = \sum_{\Sigma m_i=0}^{n} \frac{m_1! m_2! \cdots m_s!}{r^{m_1+m_2+\cdots+m_s}} \alpha_{m_1,\cdots m_s} \exp \omega$$

avec $\omega = -i(m_1\theta_1 + m_2\theta_2 + \cdots + m_s\theta_s)$, et tel que, pour toute solution holomorphe de l'équation (a), dans (D), on ait

$$\frac{1}{(2\pi)^s}\int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} S_n(\theta_1, \cdots, \theta_s, r) \times f(x_1+re^{i\theta_1}, \cdots, x_s+re^{i\theta_s})\, d\theta_s = 0$$

quel que soit $r\neq 0$, de sorte que le point $(x_1+re^{i\theta_1}, \cdots, x_s+re^{i\theta_s})$ ne sorte pas de (D). Si S_n se transforme en une série de Fourier, l'équation $E_n=0$ devient une équation aux dérivées partielles d'ordre infini. On montre ensuite que, inversement, les solutions holomorphes dans (D) de l'équation d'ordre infini, telle qu'elle vient d'être définie, vérifient la relation intégrale, ce qui permet de conclure que cette relation caractérise les solutions holomorphes de l'équation E_∞. On considère quelques équations aux dérivées partielles d'ordre infini déduites de la série de Taylor et l'on intègre.

Dans le chapitre 11, l'auteur considère une courbe fermée plane, γ paramétriquement définie par deux séries de Fourier, $\varphi(\theta)$ et $\psi(\theta)$ uniformément convergentes, et l'intégrale

$$I(x, y, \gamma) = \int_0^{2\pi} (f[x+\varphi(\theta), y+\psi(\theta)]d\theta$$

avec $f(x, y)$ analytique, dans un domaine (D) qui contient (γ). Pour les solutions analytiques de l'équation aux dérivées partielles, d'ordre infini,

$$E_\infty(f) = \sum_{n=1}^{\infty} \left[\frac{1}{n!} \sum_{i,j=0}^{n} \alpha_{ij} \frac{\partial^{i+j} f}{\partial x^i \partial y^j}\right]$$

on a $(2\pi)^{-1} I(x, y; \gamma) = f(x, y)$. Si l'on considère E_∞ comme limite de la suite des équations E_n, avec

$$E_n = \sum_{k=i}^{n} \left[\frac{1}{k!} \sum_{i,j=0}^{k} \alpha_{ij} \frac{\partial^{i+j} f}{\partial x^i \partial y^j}\right],$$

on trouve que, pour tout $\varepsilon>0$, on peut déterminer un rang N tel que pour toutes les solutions analytiques $f(x, y)$ de l'équation aux dérivées partielles $E_n=0$ $(n\geq N)$, l'on ait:

$$\left|\frac{1}{2\pi} I(x, y, \gamma) - f(x, y)\right| < \varepsilon.$$

On détermine ensuite les courbes (γ) pour lesquelles $ds=K(\theta)d\theta$, $K(\theta)$ étant un polynôme trigonométrique, et l'on obtient ainsi les formules de moyenne du type de Gauss (affectées de masses unités).

Dans le dernier chapitre, l'auteur étudie les conditions permettant d'obtenir une formule du type Saks-Green, donnée par le théorème S$_2$". (Résumé de l'auteur)

R. P. Boas, Jr. (Evanston, Ill.)

2545:

Cattabriga, Lamberto. Su alcuni problemi per equazioni differenziali di tipo composito. Rend. Sem. Mat. Univ. Padova 27 (1957), 122–143.

Partial differential equations are said to be of "composite type" if the algebraic expression used to determine type has one or more linear factors plus one or more irreducible factors of higher degree, real numbers being used. The composite equation $(\partial/\partial x)\Delta u=0$, where Δ denotes the Laplacian $(\partial^2/\partial x^2)+(\partial^2/\partial y^2)$, has been considered by Sjöstrand [Ark. Mat. Astr. Fys. **25A** no. 21 (1936), 1–11; **26A** no. 1 (1937), 1–10], Hadamard [Tohoku Math. J. 37 (1933), 133–150], and the reviewer [Proc. Amer. Math. Soc. 3 (1952), 751–756; 5 (1954), 720–725; MR 14, 382; 16, 260].

In the present paper the author proves existence and uniqueness of a function $u(x, y)$ satisfying the above equation inside the unit circle, and assuming prescribed "boundary" values on the circle $x^2+y^2=1$, and also on the y-axis: $-1\leq y\leq 1$, $x=0$. The hypothesis requires that the boundary data satisfy a Hölder condition.

The method utilizes Hadamard's sum representation, and Sjöstrand's Poisson integral approach, to obtain a Fredholm integral equation, which is then handled by using new (and stronger) estimates. A number of similar problems for various higher-order composite equations are also treated.

R. B. Davis (Syracuse, N.Y.)

POTENTIAL THEORY

See also 2532, 2544.

2546:

Itô, Jun-iti. Asymptotic properties of subharmonic and analytic functions. Proc. Amer. Math. Soc. 9 (1958), 763–772.

The author is concerned with establishing generalisations and refinements of results of Boas [Duke Math. J. **20** (1953), 433–448; MR **15**, 517]. The work of Boas to which reference is made treats the problem of the extent to which the boundary conditions of the Ahlfors-Heins theorem [Ann. of Math. (2) **50** (1949), 341–346; MR **10**, 522] may be weakened for functions of exponential type. The present paper uses the Riesz representation theorem

for subharmonic functions, a substitute of Carleman's theorem for subharmonic functions, and the methods developed by Boas [loc. cit.]. Typical of the results of the present paper is the following theorem. If $u(z)$ is subharmonic for Re $z \geqq 0$ and satisfies

$$\liminf_{r\to\infty} r^{-1}\int_{-\pi/2}^{\pi/2} u^+(re^{i\theta})d\theta < +\infty,$$

and for each $\varepsilon > 0$, there exists N such that

$$\int_{r_1}^{r_2} t^{-2}u(\pm it)dt < \varepsilon \quad (r_2 > r_1 > N),$$

then $\lim_{r\to\infty} u(re^{i\theta})/r\cos\theta$ exists (finite) save for exceptional sets of exactly the same kinds as those of the Ahlfors-Heins theorem. *M. H. Heins* (Urbana, Ill.)

2547:

Dinghas, Alexander. Wachstumsprobleme harmonischer und verwandter Funktionen in E^n. Ann. Acad. Sci. Fenn. Ser. A. I. no. **250/8** (1958), 14 pp.

The author presents a generalization of the Phragmén-Lindelöf principle which extends earlier work due to himself and to A. Huber. Let u be a harmonic function with non-positive boundary values in an infinite region G of n-dimensional space. Let S_r be the intersection of the surface of the sphere of radius r about a suitable origin with the portion of G where $u > 0$, and let $\mu(r)$ be the integral of u^2 over S_r with respect to the $(n-1)$-dimensional area element on the unit sphere. By means of a familiar eigenvalue problem, the author introduces a function $\psi(r)$ depending only on the geometry of G such that $[\mu(r)r^{n-2}-\mu(r_0)r_0^{n-2}]/\psi(r)$ is increasing and such that $\lim_{r\to\infty}\mu(r)r^{n-2}/\psi(r)=\alpha>0$ exists.

P. R. Garabedian (Stanford, Calif.)

2548:

Delsarte, Jean. Note sur une propriété nouvelle des fonctions harmoniques. C. R. Acad. Sci. Paris **246** (1958), 1358–1360.

The author sketches a proof for the following theorem: Let f be a function which is defined and continuous in R^n, $n>2$, and let $F(x, r)$ denote the mean value of the function f taken over a sphere with center at x and radius r. Further let a and b be two fixed real numbers which are positive and distinct. If $F(x, a)=F(x, b)=f(x)$ for each x in R^n, then the function f is harmonic in R^n. When $n>3$, the proof fails if a/b assumes certain exceptional values. These values are finite in number and they are independent of the function f. *F. W. Gehring* (Helsinki)

2549:

Kishi, Masanori. Inferior limit of a sequence of potentials. Proc. Japan Acad. 33 (1957), 314–319.

L'auteur donne diverses extensions du théorème de convergence établi pour les suites de potentiels newtoniens par Brelot et Cartan [voir H. Cartan, Bull. Soc. Math. France 73 (1945), 74–106; MR **7**, 447]. L'une d'elles, concernant les noyaux réguliers (i.e., satisfaisant au principe de continuité de Evans-Vasilesco), a été énoncée indépendamment par G. Choquet [C. R. Acad. Sci. Paris **244** (1957), 1606–1609; MR **19**, 405]. Voir aussi l'article récent de M. Brelot et G. Choquet [J. Madras Univ. Sect. B. **27** (1957), 277–286; MR **19**, 261]. *J. Deny* (Strasbourg)

2550:

Kishi, Masanori. Capacities of borelian sets and the continuity of potentials. Nagoya Math. J. **12** (1957), 195–219.

Let Ω be a locally compact metrisable space, μ a non-negative Radon measure with compact carrier S_μ in Ω and $\Phi(P, Q)$ a positive symmetric real-valued function on $\Omega\times\Omega$ which is continuous in the wider sense and finite for different P and Q. The potential $U^\mu(P)$ is defined by $\int\Phi(P, Q)d\mu(Q)$. The capacity $c(K)$ of a compact set K is defined by $\sup\mu(\Omega)$ for μ with $S_\mu\subset K$ such that $U^\mu(P)\leqq 1$ in Ω, the inner capacity $\mathrm{cap}_i(A)$ of any set A by $\sup c(K)$ for compact $K\subset A$ and the outer capacity $\mathrm{cap}_e(A)$ by $\inf \mathrm{cap}_i c(G)$ for open $G\supset A$. A property is said to hold nearly everywhere on a set if the inner capacity of the exceptional set is zero. If $\mathrm{cap}_i(A)=\mathrm{cap}_e(A)$, A is called capacitable. Choquet [Ann. Inst. Fourier, Grenoble **5** (1953–54), 131–295 (1955); MR **18**, 295] proved the capacitability of any analytic set for Newtonian potentials. The main result of the paper under review is the capacitability of any analytic set relatively compact in Ω under the assumption that there exists an equilibrium measure μ_k on any compact set K; this implies that $S_{\mu_k}\subset K$, $\mu_k(\Omega)=\mathrm{cap}_i(K)$, $U^{\mu_k}(P)\leqq 1$ in Ω and $U^{\mu_k}(P)=1$ nearly everywhere on K. In the last section the author discusses a function $m^*(P, U^\mu)$ defined as follows: For a potential U^μ and an open set $\omega=\omega(P)\ni P$ we define $m^*(U^\mu, \omega(P))$ by the supremum of numbers c for which $\mathrm{cap}_i(\{Q\in\omega;\ U^\mu(Q)<c\})=0$, and set $m^*(P, U^\mu)=\sup_\omega m^*(U^\mu, \omega(P))$. {A paper by Aronszajn and Smith [ibid. **6** (1955–56), 125–185; MR **18**, 319] may better be added to the bibliography.} *M. Ohtsuka* (Hiroshima)

2551:

Pfluger, Albert. Harmonische und analytische Differentiale auf Riemannschen Flächen. Ann. Acad. Sci. Fenn. Ser. A. I, no. **249/4** (1958), 18 pp.

Für das Bestreben, nicht-kompakte Riemannsche Flächen durch Eigenschaften der Räume der auf ihnen harmonischen Differentiale zu klassifizieren, ist es zweckmäßig, zunächst solche Eigenschaften dieser Räume zu finden, die für alle Riemannschen Flächen gelten. Zu diesem Zwecke entwickelt Verfasser die Cohomologietheorie für differenzierbare Flächen und spezialisiert sie — unter natürlicher Modifizierung — für Riemannsche Flächen auf den durch die konforme Struktur ausgezeichneten Hilbert-Raum der quadrat-integrierbaren harmonischen Differentiale, in welchem es das Dirichletsche Prinzip gestattet, Cohomologieaussagen zu Orthogonalitätsbeziehungen zu verschärfen. — Die in Verfassers Monographie "Theorie der Riemannschen Flächen" [Springer, Berlin-Göttingen-Heidelberg, 1957; MR **18**, 796] schon im wesentlichen enthaltene Theorie wird mit der vorliegenden Arbeit in bestechender Weise übersichtlich dargestellt und diskutiert. *H. Tietz* (Münster)

SEQUENCES, SERIES, SUMMABILITY

See also 2404.

2552:

Thron, W. J. Convergence of infinite exponentials with complex elements. Proc. Amer. Math. Soc. **8** (1957), 1040–1043.

The author uses methods similar to those employed in convergence region and value region problems for continued fractions to prove the following theorem on infinite exponentials: If $|a_n|\leqq e^{-1}$ $(n=1, 2, \cdots)$, then the sequence

$\{T_n(1)\}$, where $T_n(z)=\exp a_1 \exp a_2 \cdots \exp a_n z$, converges to a value u which satisfies $|\log u|\leq 1$.

W. T. Scott (Evanston, Ill.)

2553:

Šalát, T. Über einen Satz von Dini. Acta Fac. Nat. Univ. Comenian. Math. 2 (1957), 67–70. (Slovak. Russian and German summaries)

Ein bekannter Satz über die Reihen mit positiven Gliedern von Dini sagt: Es sei $\sum_{n=1}^{\infty} C_n$ eine konvergente Reihe mit positiven Gliedern, $r_k=\sum_{n=1}^{\infty} C_{n+k}$ $(k=1, 2, \cdots)$, und $r_0=\sum_{n=1}^{\infty} C_n$. Dann ist die Reihe $\sum_{n=1}^{\infty} (C_n/r_{n-1}{}^{\alpha})$ für $\alpha<1$ konvergent und für $\alpha\geq 1$ divergent.

In dieser Arbeit studiert man "die Schnelligkeit" der Divergenz von $\sum_{n=1}^{\infty} (C_n/r_{n-1})$ und das Hauptergebnis der Arbeit ist der Satz:

Es sei $1=\sum_{n=1}^{\infty} C_n=r_0$ und $r_k=\sum_{n=1}^{\infty} C_{n+k}$ $(k=1, 2, \cdots)$. Es sei $C_n/r_{n-1}\to 0$. Dann gilt: $\sum_{k=1}^{n} (C_k/r_{k-1})\sim \log r_n^{-1}$. (Dabei $f(x)\sim g(x)$ bedeutet wie gewöhnlich: $\lim_{x\to\infty} (f(x)/g(x))=1$).

Eine einfache Folgerung dieses Satzes ist der Beweis der berühmten Beziehung: $1+\frac{1}{2}+\frac{1}{3}+\cdots+1/n\sim\log n$.

Zusammenfassung des Autors

2554:

Šalát, T. Über einige Eigenschaften der Reihen mit positiven Gliedern. Acta Fac. Nat. Univ. Comenian. Math. 2 (1957), 71–76. (Slovak. Russian and German summaries)

Es sei

$$(1) \qquad \sum_{n=1}^{\infty} a_n=a_1+a_2+a_3+\cdots+a_n+\cdots$$

eine konvergente Reihe mit positiven Gliedern. Im ersten Teil dieser Arbeit studiert man das Maß $\mu(W)$ der Menge W aller derjenigen Zahlen w, welche die Form $w=\sum_{n=1}^{\infty} \varepsilon_n a_n$ $(\varepsilon_n=+1$ oder $\varepsilon_n=-1)$ besitzen. Das Hauptergebnis dieses Teiles der Arbeit ist der Satz:

a) Es sei für $k=1, 2, 3, \cdots$ in der Reihe (1) die Bedingung $a_k>R_k$ erfüllt. (Dabei bedeutet R_k den Rest nach den k-tem Glied in (1).) Dann gilt: $\mu(W)=\lim_{n\to\infty} 2^{n+1}\cdot R_n$.

b) Es sei für $k=1, 2, 3, \cdots$ in der Reihe (1) die Bedingung $a_k\leq R_k$ erfüllt. Dann gilt: $\mu(W)=2\sum_{n=1}^{\infty} a_n$.

Im zweiten Teil der Arbeit studiert man eine Frage, welche in Zusammenhang mit divergenten Reihen steht. Der folgende Satz ist bewiesen: Es sei

$$(2) \qquad \{\varepsilon_n\}_{n=1}^{\infty}=\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots, \varepsilon_n, \cdots$$

eine Folge mit Gliedern $\varepsilon_n=+1$ oder $\varepsilon_n=-1$. Es sei $\sum_{n=1}^{\infty} a_n=+\infty$, $a_n>0$, $a_n\to 0$. Behauptung: Es existiert eine Menge der Folgen (2), welche die Mächtigkeit des Kontinuums hat, daß für jede Folge $\{\varepsilon_n\}_{n=1}^{\infty}$ aus dieser Menge $\sum_{n=1}^{\infty} \varepsilon_n a_n=+\infty$ gilt.

Ein ähnlicher Satz auch für $-\infty$ gilt. Diese Sätze sind eine Ergänzung der früheren Ergebnissen des Verfassers.

Zusammenfassung des Autors

2555:

Srivastava, Pramila. On strong Rieszian summability of infinite series. Proc. Nat. Inst. Sci. India. Part A. 23 (1957), 58–71.

Let $0<\lambda_0<\lambda_1<\cdots$, let $\lambda_n\to\infty$, and let $k\geq 0$. The Riesz transform $C_\lambda^{(k)}(x)$ of type λ_n and order k of a series $a_0+a_1+\cdots$ is then defined by

$$C_\lambda^{(k)}(x)=\sum_{\lambda_n<x}\left(1-\frac{\lambda_n}{x}\right)^k a_n.$$

If $C_\lambda^{(k)}(x)\to s$ as $x\to\infty$, the series is said to be evaluable (R, λ_n, k) to s. If $C_\lambda^{(k)}(x)$ has bounded variation over some interval $x\geq h$, the series is said to be absolutely evaluable (R, λ_n, k) or evaluable $|R, \lambda_n, k|$. If

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x |C_\lambda^{(k-1)}(t)-s|^q dt=0,$$

then $\sum a_n$ is said to be strongly evaluable to s by the Riesz method of type λ_n, order k, and exponent q or, briefly, evaluable $[R, \lambda_n, k, q]$ to s. Several theorems involving these concepts are proved. Some of them are extensions of known theorems involving the special case in which $\lambda_n=n$. Most of them are analogous to known results involving Cesàro transforms of series.

R. P. Agnew (Ithaca, N.Y.)

2556:

Gopalakrishna, J.; and Ramamohana Rao, C. Some generalized Tauberian type theorems. J. London Math. Soc. 33 (1958), 147–156.

For a monotonic increasing function g, define a transformation $F\to F^*$ by $F^*(x)=(1/g(x))\int_0^x F(t)dg(t)$. Regarding this as a summation method, the authors are interested in relationships between the limits of oscillation of F, and those of F^*; denoting lim inf $F(x)$ and lim sup $F(x)$ by l and L, and for F^*, l^* and L^*, one has $l\leq l^*\leq L^*\leq L$. When $f(x)=g(x)F(x)$ is restricted, one can expect inequalities acting in the opposite direction. Extending a theorem of the reviewer [J. Indian Math. Soc. (N.S.) **16** (1952), 147–149; MR **14**, 631], Lakshminarasimhan [ibid. **17** (1953), 55–58; MR **15**, 114] showed that there were functions A and B such that $L^*\geq A(l, L)$ and $l^*\leq B(l, L)$, provided f was non-decreasing and $g(x)$ behaved essentially like x^c. The present authors remove the restrictions on g, and allow f to be slowly oscillating, relative to g; they obtain mutual restrictions on the possible values of the four numbers l, L, l^* and L^*. As an application, they obtain an interesting theorem for "generalized" primes. Let $\{n_j\}$ increase, and define two functions ϑ and ψ by: $\vartheta(x)=\sum_{n_k\leq x} a(n_k)$, $\psi(x)=\sum_{n_k\leq x} a(n_k)/f(n_k)$ where $a(t)$ and $f(t)$ are positive, and $f(t)\to\infty$ as $t\to\infty$. Then, if $\psi(x)=\log f(x)+O(1)$, it follows that for some positive b and B, $0<b\leq \liminf \vartheta(x)/f(x)\leq 1\leq \limsup \vartheta(x)/f(x)\leq B<\infty$. In the special case $a(t)=\log t$, $f(t)=t$, this is an analogue for Chebysheff's inequalities. {In this case, however, the result can be established directly without resort to the authors' Tauberian theorem.}

R. C. Buck (Stanford, Calif.)

2557:

Myrberg, P. J. Eine Verallgemeinerung des arithmetisch-geometrischen Mittels. Ann. Acad. Sci. Fenn. Ser. A. I, no. 253 (1958), 19 pp.

φ, ψ seien rationale Funktionen zweier komplexer Veränderlichen. Das Problem, wann die Folgen $x_n=\varphi(x_{n-1}, y_{n-1})$, $y_n=\psi(x_{n-1}, y_{n-1})$, x_0, y_0 beliebig, gegen einen gemeinsamen Grenzwert konvergieren, wird behandelt unter der Annahme, daß φ und ψ analytisch, für reelle Argumente reell und monoton wachsend sind und die Eigenschaften $\varphi(kx, ky)=k\varphi(x, y)$; $\varphi(y, x)=\varphi(x, y)$; $\varphi(x, x)=x$ besitzen. Es wird durchgeführt für das arithmetisch-harmonische Mittel und für den Fall, daß φ und ψ Funktionen zweiten Grades sind: in beiden Fällen existiert der gemeinsame Grenzwert stets, außer höchstens für $y/x<0$; zum Schluß wird ein schnell konvergentes Produkt zur Berechnung des Grenzwertes hergeleitet.

H. Tietz (Münster)

APPROXIMATIONS AND EXPANSIONS

See also 2309, 2321, 2575, 2576, 2603, 2604.

2558:

Stancu, D. D. The generalization of certain interpolation formulae for the functions of many variables. Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), no. 1–2, 31–38. (Romanian. Russian and English summaries)

Newton's interpolation formula for a function of two variables involves nodes located on a rectangular lattice. The author takes as his point of departure a generalization by Steffensen in which the number of nodes used on each ordinate may vary from one abscissa to the next, and writes a formula in which both the location and the number of nodes may vary from one abscissa to the next. This formula he extends to functions of three variables, then specializes, and expresses the remainder in the more specialized formula in terms of derivatives.

R. P. Boas, Jr. (Evanston, Ill.)

2559:

Quigley, Frank. Approximation by the translates of a single function. Proc. Amer. Math. Soc. **8** (1957), 1021–1023.

The author establishes an analogue for continuous functions on certain locally compact spaces X of a theorem of Seidel and Walsh [Bull. Amer. Math. Soc. **47** (1941), 916–920; MR **4**, 10] for holomorphic functions of one complex variable. It is assumed that there exist sequences (C_n) of compacts in X and (σ_n) of homeomorphisms of X satisfying the two conditions: Ia. $K\cap C_n=\emptyset$ and Ib. $K\subset\sigma_n(C_n)$, for each compact $K\subset X$ and almost all n. Let $C(X)$ be the space of real or complex continuous functions on X, equipped with the compact open topology. Theorem: If X satisfies Ia. and Ib., to each separable subset $\mathfrak{Y}$ of $C(X)$ corresponds an $F\in C(X)$ such that $\mathfrak{Y}$ is contained in the closure of the set of "translates" $F\circ\sigma_n^{-1}$. The proof is ingenious and uses only simple properties of locally compact spaces and Urysohn's Lemma. Alternative hypotheses are given, as is also an analogue for differentiable functions on (real) differentiable manifolds.

R. E. Edwards (Woking)

2560:

Zamfirescu, Ion. Une généralisation du théorème de Weierstrass-Stone. C. R. Acad. Sci. Paris **246** (1958), 524–525.

Let R be a topological space, V a locally convex space. $R(V)$ denotes the topological vector space of continuous maps $R\to V$, the topology being that of convergence uniform on compact sets $K\subset R$; $L(V)=V'$, the dual of V, is topologised similarly (subspace of the set of continuous maps $V\to$reals). A subset M of $L(V)$ is called an approximative base if it is linearly independent and total in $L(V)$. Théorème I gives an extension of the Weierstrass-Stone Theorem: If $U\subset R(V)$ satisfies the following conditions 1)–4), it is dense in $R(V)$. 1) U is a vector subspace of $R(V)$; 2) U separates points of R; 3) if $f,g\in U$, there exists $h\in U$ such that $\Gamma(f)\cdot\Gamma(g)=\Gamma(h)$ for all $\Gamma\in M$; 4) U contains the constant functions. If R is a topological vector space or a topological group, the theorem applies to the set U of polynomial maps $R\to V$.

An analogous theorem is given for subsets of the set $R^0(V)$ of many-valued continuous maps $R\to V$, the hypotheses now involving the partial order on $R^0(V)$ defined by $f'\leq f''$ if and only if $f'(x)\subset f''(x)$ $(x\in R)$.

Proofs are to appear elsewhere.

R. E. Edwards (Woking)

2561:

van der Sluis, A. Orthogonal polynomials and hypergeometric series. Canad. J. Math. **10** (1958), 592–612.

(I) The author presents a theory of Padé approximants for Laurent series. In particular, "regular" approximants are studied, recurrence relations for the approximants are given, and orthogonality relations of the approximants are found. It is also indicated how this theory can be extended to matrix polynomials. (II) In order to derive certain special types of orthogonal polynomials, the author generalizes a result of Padé [Ann. Sci. Ecole Norm. Sup. (3) **24** (1907), 341–400]. Explicit expressions for Padé approximants of Gauss and Heine series are given. The resulting polynomials are the classical ones and basic analogues of them [cf. also Hahn, Math. Nachr. **2** (1949), 4–34; MR **11**, 29]. (III) It is proved that "under a much more natural and apparently less restrictive condition, no more general polynomials result than those obtained" in II.

E. Frank (Chicago, Ill.)

2562:

Thron, W. J. On parabolic convergence regions for continued fractions. Math. Z. **69** (1958), 173–182.

The author continues his study of parabolic convergence regions for continued fractions $a_1/1+a_2/1+\cdots$. There is obtained an estimate of the error committed if the continued fraction is replaced by one of its approximants, a result which also gives information concerning the factors that influence the rapidity of convergence. A convergence neighborhood is found in which the convergence is proved uniform, even though the elements a_n are functions of any number of variables. Among other convergence theorems proved here, two results are established in which the a_n lie in different parabolas for different values of n.

E. Frank (Chicago, Ill.)

FOURIER ANALYSIS

See also 2828, 2829.

2563a:

Tanimura, Masayoshi. On the solution of some mixed boundary problems. VII. An integral equivalent to an extension of the Fourier series. Tech. Rep. Osaka Univ. **7** (1957), 297–306.

2563b:

Tanimura, Masayoshi. On the solution of some mixed boundary problems. VIII. The integral representation of an extension of the Fourier-Bessel-Dini series. Tech. Rep. Osaka Univ. **7** (1957), 307–313.

[For parts I–IV see same Rep. **5** (1955), 77–102, 337–348; **6** (1956), 63–74; 265–271; **7** (1957), 47–53, 55–63; MR **17**, 629; **18**, 581; **20**#5344 a-c.]

Two common cases of generalized Fourier series representations of functions $f(x)$ $(x_1<x<x_2)$ are replaced by representations by integrals in the complex plane, where the integrands involve the real parameter x. A formal evaluation of the integrals by residues reduces the integrals to the series. The author claims that the integral representations are useful in solving certain boundary value problems in partial differential equations containing a boundary condition of the type $h(x)u(x,b)+u_y(x,b)=f(x)$ on a boundary $y=b$ $(x_1<x<x_2)$, where $h(x)$ is a given

step function. He gives one example, but he does not complete its solution and his procedure is not made clear.
R. V. Churchill (Ann Arbor, Mich.)

2564:
Miller, G. F. Summation of a slowly convergent Fourier series occurring in a fluid motion problem. Proc. Roy. Soc. London. Ser. A. **237** (1956), 17–27.

2565:
Gosselin, Richard P. On the divergence of Fourier series. Proc. Amer. Math. Soc. **9** (1958), 278–282.

The author generalizes Kolmogoroff's example of an integrable function with an almost everywhere divergent Fourier series. He proves the following theorem. Let $\{p_n\}$ be a sequence of integers increasing to ∞; then there is an integrable function f such that the sequence of partial sums $s_{p_n}(x; f)$ diverges almost everywhere.
A. Shields (Ann Arbor, Mich.)

2566:
Kumari, Sulaxana. On the nonconvergence of Fourier series. Proc. Amer. Math. Soc. **9** (1958), 293–299.

The author proves four theorems on the divergence at a point of a Fourier series. Typical are the following. Theorem 1: There exists an even, integrable, periodic function $\phi(t)$ for which $\int_0^t \phi(u)du=O(t^k)$ as $t\to 0$ $(k>1)$, and $a_n=O(n^{-1/k})$, and whose Fourier series diverges at $t=0$. Theorem 2: There exists an even, integrable, periodic function $\phi(t)$ for which $\int_0^t \phi(u)du=o(t^k)$ as $t\to 0$ $(k>1)$, and $a_n=O(n^{-1/k}\rho_n)$, $\{\rho_n\}$ being any arbitrarily chosen sequence of numbers tending to ∞, and whose Fourier series diverges at $t=0$. The remaining two theorems deal with the case $k=1$.
A. Shields (Ann Arbor, Mich.)

2567:
Hsiang, Fu Cheng. On Riesz summability of Fourier series. Proc. Amer. Math. Soc. **9** (1958), 37–44.

Hardy and Littlewood [Ann. Scuola Norm. Sup. Pisa, (2) **3** (1934), 1–20] gave a criterion for the convergence of a Fourier series of a function at a point, with two assumptions. One of them governs the mean continuity of the function at that point, while the other is a Tauberian condition on the Fourier coefficients. The first yields Riesz summability of a suitable order and type, and the second then ensures convergence. The author here generalizes the Hardy-Littlewood criterion, with a weaker assumption of continuity and with a one-sided Tauberian condition.
K. Chandrasekharan (Bombay)

2568:
Rath, P. C.; and Mohanty, R. On the convergence and summability of a series associated with the derived Fourier series. Proc. Amer. Math. Soc. **9** (1958), 11–17.

If $\sum_1^\infty B_n(x)$ is the conjugate series of the Fourier series of an integrable function, the derived Fourier series is $\sum_1^\infty nB_n(x)$. The authors consider the convergence and summability of the series $\sum_1^\infty n^{-1}S_n(x)$, where $S_n(x)=\sum_{r=1}^n rB_r(x)$, and show that the behaviour is similar to that of $\sum_1^\infty B_n(x)$. As the authors admit, the results obtained here are simple Tauberian consequences of known results.
K. Chandrasekharan (Bombay)

2569:
Singh, Basudeo. On a sequence of Fourier coefficients. Proc. Amer. Math. Soc. **7** (1956), 796–803.

Let $B_n(x)$ be the nth term of the conjugate Fourier series of $f(x)$, and let $\psi(t)=f(x+t)-f(x-t)-l$. The author shows that if

$$\Psi(t)=\int_0^t \psi(u)du=o(t) \text{ and} \int_\varepsilon^\delta \frac{|\psi(t+\varepsilon)-\psi(t)|}{t}\,dt\to 0$$

as $\varepsilon\to 0$ for some fixed δ, then the sequence $\{nB_n(x)\}$ is summable $(C, 1)$ to the value l/π. The method of proof is very near to that of the $(C, 1)$ summability of the derived Fourier series [cf. A. Zygmund, Bull. Acad. Polon. **1925**, 207–217 (1926)]. From this he derives Lebesgue's test for convergence of the conjugate Fourier series by applying Tauber's theorem.
G. Sunouchi (Zbl **72** (1958), 60)

2570:
Satô, Masako. Fourier series. XVIII. On a sequence of Fourier coefficients. Proc. Japan Acad. **33** (1957), 380–385.

Let $f\in L$ in $\langle 0, 2\pi\rangle$ with Fourier coefficients a_k, b_k, and let $B_n(x)=b_n\cos nx-a_n\sin nx$ while $\sigma_n(x)$ denotes the nth $(C, 1)$-sum of the $nB_n(x)$. Suppose that $l=f(x+0)-f(x-0)$ exists. In 1913, L. Fejér [J. Reine Angew. Math. **142** (1913), 165–188] proved that $nB_n(x)\to l/\pi$ in the (C, r)-sense for all $r>1$. B. Singh [review above] has given conditions under which this result holds for $r=1$. The author has results of the following kind. Let $0\leqq\alpha\leqq 1$, and put $\psi_x(t)=f(x+t)-f(x-t)-l$. If $\Psi_x(t)=\int_0^t \psi_x(u)du=o(t(\log 1/t)^\alpha)$, and if $\int_0^t (\psi_x(\xi+u)-\psi_x(\xi-u))du=o(t(\log 1/t)^{\alpha-1})$, uniformly in ξ, then $\sigma_n(x)-l/\pi=o(\log n)^\alpha$. Note the case $\alpha=0$.
W. W. Rogosinski (Newcastle-upon-Tyne)

2571:
Konyuškov, A. A. Best approximations by trigonometric polynomials and Fourier coefficients. Mat. Sb. N.S. **44(86)** (1958), 53–84. (Russian)

Relations between the degree of approximation $E_n(f)_p$ of a function f on $(-\pi, +\pi)$ (in the metric of the space L^p) by trigonometric polynomials for different p and its Fourier coefficients ared iscussed. Let $f(x)\sim\sum b_n\sin nx$; then

(1) $E_n(f)_q\leqq$
$$A[E_n(f)_p(n+1)^{1/p-1/q}+\textstyle\sum_{k=n+1}^\infty E_k(f)_p\,k^{1/p-1/q-1}]$$
for $p<q\leqq+\infty$;

(2) $\sum_{k=2n}^\infty k^{-2}b_k\leqq An^{-2+1/p}E_n(f)_p$ if $b_n\geqq 0$;

(3) $b_{3n}\leqq An^{1/p-1}E_n(f)_p$ if $n^{-\tau}b_n$ is almost decreasing for some $\tau>0$ (a sequence $c_n\geqq 0$ is almost decreasing if $c_m\leqq A'c_n$ for $m\geqq n$);

(4) $E_n(f)_p\leqq A[b_{n+1}(n+1)^{1/p'}+(\sum_{k=n+1}^\infty b_k{}^pk^{p-2})^{1/p}]$

if $b_n\downarrow 0$; the last term in $[\cdots]$ may be replaced by $\sum_{k=n+1}^\infty b_nk^{-1/p}$ if $n^{-\tau}b_n\downarrow 0$ for some $\tau>0$. Here A, A' denote some constants, and p' is the conjugate exponent to p. Further results include the equivalence of relations such as $E_n(f)_p=O(\varphi_n)$, $\sum_{k=n+1}^\infty b_k{}^{p'}=O(\varphi_n{}^{p'})$, $b_n=O(n^{1/p-1}\varphi_n)$ and $E_n(f)_q=O(n^{1/p-1/q}\varphi_n)$, if $n^{-\tau}b_n\downarrow 0$ for some $\tau>0$, $1<q<p<+\infty$, and the sequence φ_n satisfies some regularity conditions; similar results with the "O" relation replaced by equivalence $\sim$; theorems that $\sum n^\gamma|b_n|^\beta<+\infty$ implies $\sum n^{(\gamma-\beta)/p'}E_n{}^\beta(f)_p<+\infty$ or is implied by it for some values of the parameters. The following inequality is used here: $\sum_{n=1}^\infty n^{\gamma-\alpha}(\sum_{k=n}^\infty d_k)^\alpha\leqq A\sum_{n=1}^\infty n^\gamma d_n$, where $0<\alpha<1$, $\gamma>\alpha-1$, and $n^{-\tau}d_n$ is almost decreasing for some $\tau>0$. These theorems include results of many authors, in particular Hardy and Littlewood, Stečkin [Izv. Akad. Nauk SSSR, Ser. Mat. **15** (1951), 219–242; Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 37–40; MR **13**, 29; **16**, 1101], Bari and Stečkin [Trudy

Moskov. Mat. Obšč. **5** (1956), 483–522; MR **18**, 303], the reviewer [Math. Z. **51** (1948), 135–149; MR **10**, 33]; their methods are often used. *G. G. Lorentz* (Syracuse, N.Y.)

2572:

Czipszer, J.; et Freud, G. Sur l'approximation d'une fonction périodique et de ses dérivées successives par un polynome trigonométrique et par ses dérivées successives. Acta Math. **99** (1958), 33–51.

Let C denote the space of continuous and 2π-periodic functions with norm $\|f\|=\max|f(x)|$. Let T_n denote the set of trigonometric polynomials of order n. For $f\in C$, let $E_n(f)=\min_{U_n\in T_n}\|f-U_n\|$. The authors show that if $f\in C$, $P_n\in T_n$, $\|f-P_n\|\leqq\varepsilon$, then the following inequalities hold: (*) $\|f^{(k)}-P_n^{(k)}\|\leqq 3\cdot 2^k n^k\varepsilon+4E_n(f^{(k)})$ if $f^{(k)}$ exists and is continuous; $\|\tilde f^{(k)}-\tilde P_n^{(k)}\|\leqq 3\cdot 2^k n^k\varepsilon+4E_n(\tilde f^{(k)})$ if $\tilde f^{(k)}$, the derivative of the conjugate function, exists and is continuous; $\|\tilde f-\tilde P_n\|\leqq A\,\varepsilon\log(n+1)+4E_n(\tilde f)$, A denoting a universal constant, if $\tilde f$ is continuous. The proof is obtained by using the de La Vallée Poussin mean, which is a combination of Fejér sums of Fourier series. Results of a somewhat simpler form are obtained if one measures the distance of the function from the trigonometric polynomial of order n relative to the best trigonometric approximation of order n. For example, it is shown that if $f\in C$, $P_n\in T_n$, $\|f-P_n\|\leqq\delta E_n(f)$, then (**) $\|f^{(k)}-P_n^{(k)}\|\leqq 12\cdot 2^k\delta E_n(f^{(k)})$, provided $f^{(k)}$ exists and is continuous. A similar result holds for $\tilde f^{(k)}$. Next a sort of localization result analogous to (*) is obtained. The remainder of the paper is devoted to obtaining improved estimates for the constants in some of the inequalities. For example, (*) is replaced by

$$\|f^{(k)}-P_n^{(k)}\|\leqq\Theta_1\log[\min(k,n)+1][n^k\varepsilon+E_n(f^{(k)})],$$

and (**) is replaced by

$$\|f^{(k)}-P_n^{(k)}\|\leqq\Theta\delta\log[\min(k,n)+1]E_n(f^{(k)}),$$

where Θ_1 and Θ denote universal constants. The proof is obtained by introducing a generalization of the de La Vallée Poussin means. There are some analogous results for $f\in L^p$. *J. G. Herriot* (Stanford, Calif.)

2573:

Eggleston, H. G. The Bohr spectrum of a bounded function. Proc. Amer. Math. Soc. **9** (1958), 328–332.

Sei F die Menge der komplexwertigen Funktionen, die in $-\infty<x<+\infty$ erklärt, beschränkt und gleichmäßig stetig sind. C. S. Herz vermutete: Ist $f\in F$, so gilt

$$\lim_{N\to\infty}(2N)^{-1}\int_{-N}^{N}\exp(-itx)f(x)dx=0$$

für fast alle reellen t. Verfasser bestätigt diese Vermutung, sogar in allgemeinerer Form. Seinen Beweis kann man übersichtlicher fassen. Wesentlich ist folgendes Prinzip. Nach üblichem Schema (Approximation und Modifikation) darf man in der Vermutung F durch die Menge G ersetzen, die aus allen Treppenfunktionen besteht, die nur endlich viele verschiedene Werte annehmen und gleichlange Stufen besitzen. Die Behandlung von G führt natürlich auf Gleichverteilungsfragen. Jedoch erscheint dem Referenten die Stelle "It is easy to deduce" auf Seite 330 oben als Lücke; die Bedingung über die Differenzen der m_k muß wohl durch eine Bedingung über die Dichte ersetzt werden. *K. Zeller* (Tübingen)

2574:

Caracosta, Georges; et Doss, Raouf. Sur l'intégrale d'une fonction presque périodique. C. R. Acad. Sci. Paris **246** (1958), 3207–3208.

For ordinary almost periodicity, the following two theorems are stated. Th. 1: In order that a function $F(x)$ can be uniformly approximated by functions which are indefinite integrals of almost periodic functions, it is necessary and sufficient that $F(x)$ is uniformly continuous, and $F(x+\delta)-F(x)$ is almost periodic for every δ. Th. 2: It is also necessary and sufficient that $F(x)$ can be written $\int_0^x f(t)dt+\varphi(x)$, where $f(x)$ and $\varphi(x)$ are almost periodic. Th. 3 and Th. 4 of the note deals with the questions that arise naturally from Th. 2, when Stepanoff and Riemann-Stepanoff almost periodicity (and corresponding limit notions) are brought into the picture. *E. Følner* (Copenhagen)

2575:

Mandelbrojt, Szolem. Fonctions analytiques et analyse harmonique. Ann. Sci. Ecole Norm. Sup. (3) **74** (1957), 1–23.

This paper is another link in a chain of papers by the author and his collaborators concerning problems centering around quasi-analyticity, general closure problems, moment problems, etc. Here, the main results are extensions of results previously obtained by the author, in some instances jointly with collaborators.

One of the principal tools used here, as in the previous papers, is the transform of Fourier-Carleman given by

$$F^+(z)=(2\pi)^{-\frac12}\int_{-\infty}^{0}f(t)e^{-itz}dt\ \ (y>0),$$

$$F^-(z)=-(2\pi)^{-\frac12}\int_{0}^{\infty}f(t)e^{-itz}dt\ \ (y<0),$$

where f belongs to some suitable class for which the integrals are defined; e.g., $f\in L$ or L^∞. If $f\in L$ and $F=F^+-F^-$, define $s(f)=[u|F(u)\neq0]$. If $f\in L$ or L^∞, $c(f)$ is the set of points on the real line each of which is contained in a neighborhood across which F^+ and F^- are analytic extensions of each other. The spectrum of f is defined as $\sigma(f)=\mathscr{C}c(f)$, the complement of $c(f)$. [For other equivalent definitions of spectrum see C. Herz, Trans. Amer. Math. Soc. **86** (1957), 506–510; MR **20** #208.]

The author generalizes his concept of "positive functions associated with a closed set E" to "positive sequences $\{M_n\}$ associated with a set $\{\mu_n\}$ of non-negative integers and a closed set E". Using this concept a typical theorem reads as follows: Let Λ be a sequence of non-negative integers, $0\in\Lambda$, $\varphi^{(n)}\in L^\infty(n\geqq0)$, $\|\varphi^{(n)}\|_\infty\leqq M_n$ and $\varphi^{(\lambda)}(0)=0$ for $\lambda\in\Lambda$. If $\{\mu_n-1\}$ is the complement of Λ relative to the non-negative integers and $\{M_n\}$ is associated with $\{\mu_n\}$ and $\sigma(\varphi)$, then $\varphi\equiv0$.

Other theorems involving mixed closure problems of the derivatives and translates of functions in L are given in terms involving explicitly the constants $\{M_n\}$ and the spectra of the functions in question. For example, if Λ is a sequence of integers with lower density greater than $\frac12$, and $f^{(n)}$, $g\in L(n\geqq0)$, then explicit conditions are given that $f(x+a)$ for any real a be in the closure, in the topology of L, of expressions of the form

$$\sum_{n=1}^{N}a_nf^{(\lambda_n)}(x)+\sum_{n=1}^{M}b_ng(x+\xi_n),\ \lambda_n\in\Lambda.$$

These conditions are in terms of the sets $s(f)$, $\mathscr{C}s(g)$, the boundary of $\sigma(f)$, the numbers $\|f^{(n)}\|_1$, and the divergence of an integral, of the type now familiar in these types of problems. These results are generalizations and refinements of results previously obtained by the author

[Rice Inst. Pamphlet, Special Issue, Houston, Texas, 1951; Ann. Soc. Polon. Math. **25** (1952), 241–251; MR **13**, 540; **14**, 1068]. Applications are made to approximations by polynomials and to moment problems.

The last section of the paper is devoted to some arithmetic properties of the spectrum. Let $S(u)$ be a positive non-decreasing function such that $\int^{\infty} S(u)/u^2\,du<\infty$. $C(S)$ shall be the class of measurable functions f such that $f(x)=O(\exp S(|x|))$ $(|x|\to\infty)$. For $f\in C(S)$ let Λ be a uniformly isolated sequence of points in $\sigma(f)$; i.e., there exists a $q>0$ such that, for all $\lambda\in\Lambda$, $(\lambda-q, \lambda+q)$ does not contain any other point of $\sigma(f)$. Such a point can be shown to be a pole of the analytic extension of the Fourier-Carleman transform (F^+, F^-). Designate the residue at λ by $a(\lambda)$. Under suitable conditions on the sequence Λ and the function f an upper bound is obtained for

$$\liminf_{\lambda\to\infty}\log(|a(\lambda)|/\lambda).$$

{Finally, the reader should note that "$\mathscr{C}s(f)$" replaces "$\overline{c(f)}$" in the following places: p. 4, Theorem V; p. 5, lines 10 and -2; p. 13, lines 5, 6, 7, 11, 12; p. 14, line 10.}

A. Devinatz (St. Louis, Mo.)

2576:

Mandelbrojt, Szolem. Quelques relations équivalentes dans la théorie constructive des fonctions. C. R. Acad. Sci. Paris **245** (1957), 1869–1871.

If f is defined on the real line and $f^{(n)}\in L$ $(n\geqq 0)$, $\delta(f)$ shall be the closure, in L, of all finite linear combinations, with complex coefficients, of these derivatives. If $g\in L$, $\tau(g)$ shall be the closure, in L, of all finite linear combinations of the translates of g, and $\delta\tau(f, g)$ shall be the closure of all finite linear combinations of derivatives of f and translates of g.

Designate by M a sequence of positive numbers $\{M_n\}$, and by $L(M)$ the set of all f such that $\|f^{(n)}\|\leqq M_n$. If E is a closed subset of the real line, let $E_a=\cup\{[x-a, x+a]; x\in E\}$. Finally, if $g(x)\exp(-\varepsilon|x|)\in L$, g has a spectrum designated by $\sigma(g)$, and if $g\in L$, $s(g)$ shall be the set where the Fourier transform of g is not zero.

The relations $R_\delta(M; E; a)$, $R_{\delta\tau}(M; E; a)$, $R_H(M; E; a)$ are defined as follows: $R_\delta(M; E; a)$, $f\in L(M)$ and $\sigma(f)\subset E_a$ implies that $\tau(f)\subset\delta(f)$; $R_{\delta\tau}(M; E; a)$, $f\in L(M)$, $g\in L$, and $\sigma(f)\cap C_1 s(g)\subset E_a$ implies that $\tau(f)\subset\delta\tau(f, g)$ ($C_1 s(g)$ is the complement of $s(g)$ with respect to the real line); $R_H(M; E; a)$ and the fact that $\Phi(z)$ is analytic in C_2E_a, with $|\Phi(z)z^n y|\leqq M_n$ $(n\geqq 0, z=x+iy)$, implies that $\Phi\equiv 0$ (C_2E_a is the complement of E_a with respect to the complex plane if E_a is not the whole line; otherwise C_2E_a is the upper half-plane $y>0$).

From previous results of the paper reviewed above it follows that every relation R_H is a relation $R_{\delta\tau}$, and the latter is a relation R_δ. This note establishes the following partial converse:

If $0\in E$, every relation $R_\delta(M; E; a)$ is a relation $R_{\delta\tau}(M; E; a')$ and a relation $R_H(M; E; a')$ for every a' with $0<a'<a$.

A. Devinatz (St. Louis, Mo.)

2577:

Kakita, Takao. Two theorems on Fourier transform. Proc. Japan Acad. **34** (1958), 22–27.

1ère partie. Le reviewer a montré [Théorie des distributions, tome II, Hermann, Paris, 1951; MR **12**, 833; chapitre VII, théorème XV] que la transformation de Fourier $\mathscr{F}$ est un isomorphisme algébrique entre $\mathscr{O}_M$ et $\mathscr{O}_C'$, transformant suites convergentes en suites convergentes et parties bornées en parties bornées; il a énoncé le fait que $\mathscr{F}$ est un isomorphisme topologique, mais sans en publier la démonstration. C'est cette démonstration que l'auteur donne ici.

2ème partie. L'auteur généralise des relations données par Plancherel [référence citée dans l'article] sur la croissance de l'image de Fourier d'une fonction à support compact. Soit T une distribution de support compact K, sur R^n. Soit χ l'indicatrice de K, définie par $\chi(\lambda)=\sup_{y\in K}\langle\lambda, y\rangle$. Soit F l'image de Fourier de T, $F(z)=\int e^{-i\langle z,y\rangle}T_y dy$, et soit h l'indicatrice de croissance de F: $h(\lambda)=\sup_\alpha \limsup_{r\to\infty} r^{-1}\log|F(\alpha-i\lambda r)|$: alors on a $\chi(\lambda)=h(\lambda)$.

L. Schwartz (Paris)

2578:

Hirschfeld, Rudi. Sur l'analyse harmonique dans les groupes localement compacts. C. R. Acad. Sci. Paris **246** (1958), 1138–1140.

The author imposes conditions on a one parameter family of open subsets of a locally compact group which are sufficient for their use in determining a mean value as a limit of integral means. Such a mean value allows the theory of the space B^p of Besicovitch almost periodic functions, including the Fourier expansion and Riesz-Fischer theorems, to be transferred to the setting of a general locally compact group. It is pointed out that the space B^p is, essentially, the standard L^p space of the compactified group.

L. H. Loomis (Cambridge, Mass.)

INTEGRAL TRANSFORMS

See also 2537, 2575.

2579:

Bergström, Harald. Über die Konvergenz von Faltungen in verschiedenen Weierstrassnormen. Math. Nachr. **18** (1958), 244–264.

The author considers a generalized Weierstrass transform $f(x)*\Psi(x/\sigma)$, where $*$ indicates convolution and Ψ is a real linear combination of distribution functions. A norm $N_\sigma[\Psi, f]=\sup_x|f(x)*\Psi(x/\sigma)|$ is defined and it is shown that in this norm the set of distribution functions is complete if and only if $\Psi(\infty)\neq 0$. Further if Ψ is sufficiently smooth then the class of finite-valued functions which are non-decreasing except at 0 and such that $f(\pm\infty)=0$ becomes a normed ring. There are also a discussion and theorems about the problem of finding when a convolution sequence of distribution functions which form a Cauchy-sequence in the norm converges weakly to a distribution function. Necessary and sufficient conditions are obtained if Ψ satisfies certain additional conditions.

J. Blackman (Syracuse, N.Y.)

2580:

Miller, J. B. Series expansions and general transforms. Proc. Cambridge Philos. Soc. **54** (1958), 358–367.

In the author's terminology, ψ is the "h-transform" of φ if (*) $\psi(x)=\int_0^\infty h(xt)\varphi(t)dt$. The equation (*) must be given various interpretations at different times; e.g., $\int_0^\infty=\int_{\to 0}^{\to\infty}$, or $\int_0^x\psi(y)dy=d/dx\int_0^\infty(h_1(xt)/t)\varphi(t)dt$, where h is the derivative of h_1 (possibly in the sense of distributions). The following general result is obtained under complicated hypotheses: If ψ is the h-transform of φ, then $x^{-1}\lim_{N\to\infty}[\sum_{n=1}^N a_n\varphi(n/x)-\int_0^N\varphi(t/x)dR(t)]$ is the k-transform of $x^{-1}\lim_{N\to\infty}[\sum_{n=1}^N a_n\psi(n/x)-\int_0^N\psi(t/x)dR(t)]$. Here h and k are kernels of a certain type of unitary transformation on L^2, and h is related to k through the a_n. The function R is determined by the a_n. There are

many interesting special cases; for example, Poisson's summation formula and a theorem of Guinand [J. London Math. Soc. **22** (1947), 14–18; MR **9**, 279]. There are some related results. *R. R. Goldberg* (Evanston, Ill.)

2581:

Vasilach, Serge. Sur une nouvelle extension d'un théorème de Phragmen. C. R. Acad. Sci. Paris **246** (1958), 676–678.

By the use of the generalized power series defined by J. G. Mikusiński [Studia Math. **12** (1951), 181–190; MR **14**, 39] a new extension is given of Phragmen's theorem. The present work supplements the author's earlier work [C. R. Acad. Sci. Paris **243** (1956), 1468–1471; MR **19**, 842] in which another extension of the same theorem was given. *H. P. Thielman* (Ames, Iowa)

2582:

Doetsch, Gustav. Über den Konvergenzbereich von Laplace-Integralen mit komplexem Integrationsweg. Math. Nachr. **18** (1958), 129–135.

The author considers the Laplace integral

$$(1)\qquad f(s)=\int_C F(t)\exp(-st)dt,$$

where C: $t=t(l)=r(l)\exp(i\phi(l))$ is an absolutely continuous curve in the complex plane, l being the arc length. It is shown that the region of absolute convergence is convex and that $\log\int_C|F(t)\exp(-st)dt|$ is convex in this region.

If it is assumed that $\phi_2-\phi_1<\pi$, where $\liminf\phi(l)=\phi_1$ and $\limsup\phi(l)=\phi_2$, and if $\int_{l_0}^\infty\exp(-\delta r(l))dl$ converges for all $\delta>0$, then convergence of (1) at s_0 implies convergence for all $s=s_0+\rho\exp(i\theta)$; $\rho>0$ and $-\frac{1}{2}\pi-\phi_1<\theta<\frac{1}{2}\pi-\phi_2$. Further, $f(s)=(s-s_0)\int_C\Psi(t)\exp(-(s-s_0)t)dt$ and this integral is absolutely convergent in the above sector. Here $\Psi(t)=\int_{t_0}^t F(\tau)\exp(-s_0\tau)d\tau$, the integral being along C. *J. Blackman* (Syracuse, N.Y.)

2583:

Edwards, D. A. On absolutely convergent Dirichlet series. Proc. Amer. Math. Soc. **8** (1957), 1067–1074.

Let $\mathfrak{A}$ be the Banach algebra of finite Borel measures on $[0,\infty)$. By a result of R. S. Phillips [Trans. Amer. Math. Soc. **71** (1951), 393–415; MR **13**, 469] an element $\alpha\in\mathfrak{A}$ with zero singular component has an inverse in $\mathfrak{A}$ if and only if $\inf_{\sigma\geq0}|\int_0^\infty e^{-s\lambda}\alpha(d\lambda)|>0$ $(s=\sigma+it)$. From this and the fact that $\mathfrak{A}$ has no zero-divisors the author establishes in a rather straightforward way a theorem on inverses of general Dirichlet series, generalising a result of E. Hewitt and the reviewer [Proc. Amer. Math. Soc. **8** (1957), 863–868; MR **19**, 851]. As the author remarks, an alternative approach is by way of methods used by R. Arens [Trans. Amer. Math. Soc. **81** (1956), 501–513; MR **17**, 1226].

The paper also contains results (whose statements are too long to quote here) on the existence and spectral properties of elements of $\mathfrak{A}$ whose Laplace-Stieltjes transforms are of the form $f(s+\bar{\sigma})$, where $f(s)$ is the transform of an element of $\mathfrak{A}$, of a certain restricted type, and $\bar{\sigma}>0$. *J. H. Williamson* (Belfast)

2584:

Vasilach, Serge. Sur un calcul opérationnel algébrique pour fonctions de deux variables. Rev. Math. Pures Appl. **2** (1957), 181–238.

Considérons l'espace vectoriel des fonctions continues dans R^n, à support limité à gauche par rapport au cone convexe $[0,\infty[^n$. On munit cet espace d'une structure d'anneau à l'aide du produit de composition. Si $n=1$, il résulte du Théorème de Titchmarsh-Crum-Dufresnoy que cet anneau est sans diviseur de zero, d'où un corps des fractions. C'est le point de départ des travaux de Mikusiński [Studia Math. **11** (1949), 41–70; MR **12**, 189].

Pour $n\geq2$, le théorème de Titchmarsh-Crum-Dufresnoy a été généralisé par le rapporteur [C. R. Acad. Sci. Paris **232** (1951), 1530–1532, 1622–1624; MR **13**, 231. Pour d'autres démonstrations, cf. Lions, J. Analyse Math. **2** (1953), 369–380; MR **15**, 307; et J. Mikusiński et C. Ryll-Nardzeswski, Studia Math. **13** (1953), 62–68; MR **15**, 408]. Ceci permet à l'A. de généraliser en dimension quelconque la théorie de Mikusiński. (L'A. suppose $n=2$ pour simplifier l'exposé.) *J. L. Lions* (Nancy)

2585:

Zygmund, A. On singular integrals. Rend. Mat. e Appl. (5) **16** (1957), 468–505.

In former papers [e.g., Acta Math. **88** (1952), 85–139; Studia Math. **14** (1954), 249–271; Amer. J. Math. **78** (1956), 289–309; MR **14**, 637; **16**, 1017; **18**, 894] A. P. Caldéron and A. Zygmund have dealt with the extension of Hilbert's operator to singular integrals in the n-dimensional space E^n $(n\geq2)$; including that for periodic functions, i.e., the conjugate functions operator. The results were applied to potential theory. In the present paper, their main results and some of the proofs are given in a systematic and concise arrangement. {Probably a complete representation of the theory, including the earlier work of other writers on the subject, in book form would be appreciated by mathematicians.}

In § 1 the author starts from the classical Riesz theory of Hilbert's operator in E^1. Then he states the results for the class L^p in E^n and gives examples for $n=1, 2, 3$. In § 2, various generalisations in E^n are discussed, including transforms of infinite sequences of numbers and conjugate functions. In §§ 3–7 proofs of results are given; an essential tool for some of them is the Fourier transform in E^n. In § 8, application is made to the theory of the potential. A short bibliographical note is added. It refers to the work of the writers who have raised the problem. *H. Kober* (Birmingham)

2586:

Griffith, James L. On the Hankel *J*-, *Y*- and *H*-transforms. Proc. Amer. Math. Soc. **9** (1958), 738–741.

Given $G(x)$, let $g_J(\xi)=\int_0^\infty xJ_\nu(\xi x)G(x)dx$ and $g_Y(\xi)=\int_0^\infty xY_\nu(\xi x)G(x)dx$. Theorem: If $x^{\frac{1}{2}}G(x)$ belongs to $L^1(0,\infty)$ and if $-\frac{1}{2}<\nu<\frac{1}{2}$, then

$$\pi^{-1}\int_{-\infty}^{\infty}\frac{|\xi|^\nu g_J(|\xi|)\operatorname{sgn}\xi}{\xi-p}\,d\xi=-|p|^\nu g_Y(|p|);$$

$$\pi^{-1}\int_{-\infty}^{\infty}\frac{|\xi|^\nu g_Y(|\xi|)}{\xi-p}\,d\xi=-|p|^\nu g_J(|p|)\operatorname{sgn}p,$$

where both integrals are Cauchy principal value integrals. A similar theorem relates the Hankel function transform and the J-transform. *N. D. Kazarinoff* (Ann Arbor, Mich.)

2587:

Griffith, James L. Addendum to my paper "On Weber transforms." J. Proc. Roy. Soc. New South Wales **91** (1957), 189.

The author clarifies a detail in the paper of the title [same J. **89** (1956), 232–248; MR **18**, 481]. {The difficulty was pointed out to the reviewer by C. Fox.} *R. P. Boas, Jr.* (Evanston, Ill.)

2588:

Zahorska, H. Über die Punktmengen der Divergenz der singulären Integrale von Riemann-integrablen Funktionen. Acta Sci. Math. Szeged **19** (1958), 5–17.

If: (a) $K(r, x, t)$ is defined for $r>0$, x in $[a, b]$, and is L-integrable in t on $[a, b]$; (b) $\lim_{r\to\infty}\int_a^b K(r, x, t)dt=1$; (c) $\int_a^b |K(r, x, t)|dt<H(x)$ in x and r; and (d) $\lim_{r\to\infty}(\int_a^{x-\delta}+\int_{x+\delta}^b)|K(r, x, t)|dt=0$; then if $f(x)$ is a Riemann integrable function $\lim_{r\to\infty}\int_a^b K(r, x, t)f(t)dt=f(x)$ for all x except a set N of zero measure which is expressible in the form $N=\sum_{k=1}^{\infty} N_k$, where N_k are disjoint G_δ sets, and meas $\bar{N}_k=0$ ($\bar{N}_k$ closure of N_k) [see Zahorska, Fund. Math. **43** (1956), 338–357; MR **18**, 885]. A type of converse is: if: (a) $K_i(r, t)$ $(i=1, \cdots, n)$ are defined for $r>0$ and t on $[a, b]$ with $a<0<b$, and such that $\int_a^b K_i(r, t)dt=1$; (b) $\lim_{r\to\infty}(\int_a^{-\delta}+\int_\delta^b)|K_i(r, t)|dt=0$; and (c) there exists $F(r, t)$ monotone nondecreasing in t on $[a, 0]$ and monotone non-increasing in t on $[0, b]$ such that $|K_i(r, t)|<F(r, t)$ with $\int_a^b F(r, t)dt$ bounded in r; then for any set N satisfying the conditions above, and sets R_i unbounded above there exists a R-integrable function $f(x)$ such that $\lim_{r\to\infty}\int_a^b K_i(r, t)f(x+t)dt=f(x)$ for all i and all x on the complement CN of N, but $\lim_{r\to\infty, r \text{ on } R_i}\int_a^b K_i(r, t)f(x+t)dt$ does not exist for any point of N. As a corollary, for any given set satisfying the conditions above, there exists a R-integrable function $f(x)$ such that the indefinite integral $\int_a^x f(t)dt$ has $f(x)$ as derivative on CN but neither right hand, left hand or symmetric derivative exists on N.

T. H. Hildebrandt (Ann Arbor, Mich.)

2589:

Stanković, Bogoljub. Inversion d'une transformation intégrale. Univ. Beogradu. Godišnjak Filozof. Fak. Novom Sadu **1** (1956), 293–312. (Serbo-Croatian. French summary)

The results were announced in Acad. Serbe Sci. Publ. Inst. Math. **10** (1956), 85–88 [MR **18**, 481].

R. P. Boas, Jr. (Evanston, Ill.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 2623, 2641, 2986.

2590:

Rybin, P. P. Particular solutions of perturbation linear integral equation. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 19, 30–34. (Russian. English summary)

A study is made of solutions $\varphi(x, \lambda)$, which tend to infinity as λ approaches zero, of the equation

$$(1)\quad \varphi(x)=f(x)+\int_a^b k(x,s)\varphi(s)ds+\lambda\int_a^b[A_0(x, s)+ A_1(x, s)\varphi(s)+A_2(x, s)\varphi^2(s)]ds,$$

where $k(x, s)$, $f(x)$, and $A_i(x, s)$ $(i=0, 1, 2)$ are given continuous functions. Such solutions are called particular solutions. Under the assumption that there exist only two characteristic functions $p(x)$ and $q(x)$ of $k(x, s)$ belonging to the characteristic constant one, the following theorems are proved. (I) Let

$$\alpha=\int_a^b f(x)q(x)ds,\quad \beta=\int_a^b\int_a^b A_2(x, s)p^2(s)q(x)dxds;$$

then there exist two particular solutions of equation (1), provided $\alpha\neq0$, $\beta\neq0$, and α and β have opposite signs. (II) If $\alpha=0$, $\beta\neq0$, there exists no particular solution of equation (1).

H. P. Thielman (Ames, Iowa)

2591:

Ovčinnikov, P. F. Extremal properties of eigenvalues of integral equations with positive definite kernel and with a non-monotonic distribution function. Ukrain. Mat. Ž. **10** (1958), no. 2, 147–159. (Russian. English summary)

Let $\delta(x)$ be a function of bounded variation and $k(x, s)$ be a positive definite function. The author considers the linear integral equation

$$\varphi(x)=\lambda\int_0^l k(x, s)\varphi(s)d\delta(s),$$

and obtains variational characterizations of the positive and negative eigenvalues. For functions of bounded variation $a(x)$ the author defines

$$I(a, \delta)=\int_0^l\left[\int_0^l k(x, s)\, da(s)\right]^2 d\delta(x),$$

$$I(a, a, \delta)=\int_0^l\int_0^l k(x, s)da(x)da(s)/I(a, \delta)=I(a, a)/I(a, \delta).$$

The positive eigenvalues are obtained by minimizing $I(a, a, \delta)$ in the class of functions of bounded variation $a(x)$ subject to the condition $I(a, \delta)>0$ and to conditions equivalent to orthogonality to the eigenfunctions already obtained. For negative eigenvalues one maximizes $I(a, a, \delta)$ in a similar manner subject to the restriction $I(a, \delta)<0$. A minimum-maximum principle and a theorem connecting the eigenvalues of related kernels are also proved.

R. R. Kemp (Kingston, Ont.)

2592:

Todor, Liviu I. Sur quelques propriétés d'un champ vectoriel. C. R. Acad. Sci. Paris **246** (1958), 1360–1363.

Let D be a three dimensional domain with boundary S consisting of a finite number of closed surfaces. Let $\bar{V}_0$ be a continuous vector field given on S. Necessary and sufficient conditions are given that there exist a vector field $\bar{V}$ in D, with boundary values $\bar{V}_0$ on S, and such that curl $\bar{V}=0$ and div $\bar{V}=0$.

R. C. MacCamy (Pittsburgh, Pa.)

2593:

Todor, Liviu I. Conditions nécessaires et suffisantes de résolubilité de l'équation intégrale de M. H. Villat. C. R. Acad. Sci. Paris **246** (1958), 1488–1490.

Let D be a three-dimensional domain with boundary S consisting of a finite number of closed surfaces. Results of the paper reviewed above are used to obtain necessary and sufficient conditions for a solution of the vector integral equation

$$\tfrac{1}{2}\bar{V}(P')-\frac{1}{4\pi}\iint_S \operatorname{grad}_{P'}\frac{1}{r_{PP'}}[\bar{n}(P)\times\bar{V}(P)]d\sigma_P=\bar{F}(P')$$

$(P'\in S)$, where $\bar{F}$ is a continuous vector field on S, and $\bar{n}$ is the interior normal to S.

R. C. MacCamy (Pittsburgh, Pa.)

2594:

Poli, L. Equations intégrales dont le noyau est une fonction de Bessel. Bul. Inst. Politech. Iași **4** (1949), 137–142.

Si l'on pose $f(x)=\int_0^\infty J_n(2(xs)^{\frac{1}{2}})g(s)ds=H_n(g)$ la formule de réciprocité de Hankel montre que $g(x)=H_n(f)$, si par exemple, $f\in L^2$, $g\in L^2$. L'auteur se sert de cette propriété pour résoudre l'équation $\varphi(x)+\lambda H_n(\varphi)=f(x)$, où $\varphi(x)$ est l'inconnue, et étend le procédé à la résolution de certaines équations intégro-différentielles du type $\sum_1^n a_{n-i}x^i\varphi^i(x)+H_n(\varphi)=f(x)$. Enfin, il utilise le calcul symbolique pour résoudre $P(x)\varphi(x)+H_n(\varphi)=f(x)$, $P(x)$ étant un polynome donné.

R. Campbell (Caen)

2595:

Lando, Yu. K. Asymptotic behavior of eigenvalues and eigenfunctions of integro-differential equations of Volterra type. Minsk. Gos. Ped. Inst. A. M. Gor'k. Uč. Zap. **7** (1957), 21–34. (Russian)

The author extends three theorems of Naĭmark [Linear differential operators, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954; MR **16**, 702; cf. § 4] concerning the equation $y^{(n)}+p_2y^{(n-1)}+\cdots+p_{n-1}y'+p_ny+\rho^ny=0$, p_i continuous on [0, 1]. He considers the equation

$$(*)\quad u^{(n)}(x)+\sum_0^{n-2} a_i(x)u^{(i)}(x) + \sum_0^{n-1}\int_0^x A_i(x,t,\rho)u^{(i)}dt+\rho^{n+1}\int_0^x u(t)dt=0,$$

where a_i is continuous on [0, 1], and $A_i(x, t, \rho)$ is an analytic function of ρ, $\rho\in T_k$, for fixed x and t, $0\leqq x, t\leqq 1$, and where $|A_i(x,t,\rho)|\leqq C|e^{\omega_1\rho t}|$, ($i=0, 1, \cdots, n-1$, and $|\rho|>N$). The region T_i ($i=0, \cdots, 2n+1$) is the sector $i\pi/n\leqq\arg(\rho+c)\leqq(i+1)\pi/n$. Theorem: In each sector T_i and for $|\rho|\geqq N$, equation (*) has a solution $u(x,\rho)=\sum_1^{n+1}c_ku_k(x,\rho)$, where $\sum_1^{n+1}c_k[1+O(\rho^{-1})]=0$. The functions u_k are regular in ρ for $|\rho|>N$, satisfy (*), and are such that

$$u_k=e^{\omega_k\rho x}[\omega_k+O(\rho^{-1})],\quad u_k^{(n-1)}=\rho^{n-1}e^{\rho\omega_k x}[\omega_k^n+O(\rho^{-1})].$$

The numbers ω_k are the $n+1$ $(n+1)$st roots of -1 enumerated so that $\Re(\rho\omega_1)\leqq\Re(\rho\omega_2)\leqq\cdots\leqq\Re(\rho\omega_{n+1})$.

Next, boundary conditions are imposed and the asymptotic behavior of the eigenvalues and eigenfunctions for the boundary value problem is determined.

N. D. Kazarinoff (Ann Arbor, Mich.)

2596:

Rizzonelli, Pieranita. Estensione del teorema del Dini sulle funzioni implicite alle equazioni integro-differenziali del tipo "Faltung". Ist. Lombardo Accad. Sci. Lett. Rend. A. **92** (1957), 117–131.

Si dimostrano dei teoremi di esistenza e di unicità per le equazioni integro-differenziali del tipo "Faltung", lineari o no, nelle stesse condizioni del classico teorema del Dini sulle funzioni implicite. *Riassunto dell'autore*

FUNCTIONAL ANALYSIS

See also 2330, 2403, 2420, 2457, 2464, 2510, 2559, 2560, 2577, 2583, 2645, 2687, 2836.

2597:

Nakano, Hidegorô. An extension theorem. Proc. Japan Acad. 33 (1957), 603–604.

The author gives a simple proof of the theorem of Ascoli and Mazur, which says: in a locally convex topological linear space, an element which is not in a closed convex set can be separated from it by a continuous linear functional. The proof is based on a generalized form of the extension theorem of Banach which was essentially stated in his previous paper [J. Fac. Sci. Univ. Tokyo. Sect. I **6** (1951), 85–131; MR **13**, 362].

I. G. Amemiya (Kingston, Ont.)

2598a:

Kasahara, Shouro. Une généralisation du théorème de Mackey. Proc. Japan Acad. **33** (1957), 134–135.

2598b:

Kasahara, Shouro. Le problème de la dualité en une forme générale dans la théorie des espaces localement convexes. Math. Japon. **4** (1956), 63–82.

Let E be a locally convex vector space, let F be a normed space and let $L(E, F)$ [resp. $\mathscr{L}(E, F)$] be the vector spaces of all linear [resp. continuous linear] mappings of E into F. Further, let $\mathscr{L}$ be a given vector subspace of $L(E, F)$ which separates the points of E and is closed under composition on the left by the elements of $\mathscr{L}(F, F)$. Given $\mathscr{L}$, it is desired to characterize the Hausdorff topologies on E which have the property that, when E is equipped with this topology, then $\mathscr{L}$ coincides with all of $\mathscr{L}(E, F)$. This and some related questions are analyzed in some detail in the second paper, and a host of results are obtained which cannot be described briefly. However, the first paper makes an announcement of the main results. *R. G. Bartle* (Urbana, Ill.)

2599:

Deprit, André. Quelques classes d'endomorphismes d'espaces localement convexes séparés. Ann. Soc. Sci. Bruxelles. Sér. I. **71** (1957), 89–101.

Let E be a locally convex topological vector space. By an endomorphism u of E is meant a linear continuous mapping of E into itself such that the induced mapping of $E/\mathrm{Ker}(u)$ on $u(E)$ is bicontinuous. Various classes of endomorphisms are considered; among these, $\alpha(E)$ where $\mathrm{Ker}(u)$ is finite-dimensional and $u(E)$ has a topological supplement in E, and $\nu(E)$ the subset of $\alpha(E)$ where there exists $p\geqq 0$ with $\mathrm{Ker}(u^p)=\mathrm{Ker}(u^{p+1})$. Dually, the class $\beta(E)$ consists of the u where $\mathrm{Ker}(u)$ has a topological supplement and $u(E)$ is a closed linear subspace of finite co-dimension; $\lambda(E)$ denotes the subset of $\beta(E)$ for which there exists $q\geqq 0$ with $u^q(E)=u^{q+1}(E)$. Set $\rho(E)=\lambda(E)\cap\nu(E)$. Properties of these various classes are derived. The sets $\nu(E)$, $\lambda(E)$ and $\rho(E)$ are characterized in the algebra of all continuous linear mappings of E into E. For example, $u\in\nu(E)$ if and only if $u=v_1+v_2$, where $v_1\in\alpha(E)$, $\mathrm{Ker}(v_1)=(0)$, v_2 has finite-dimensional range, and $v_1v_2=v_2v_1$. *B. Yood* (New Haven, Conn.)

2600:

Lánský, Miloš. On the transformation GW. Apl. Mat. **2** (1957), 444–468. (Czech. Russian and English summaries)

The problem of chemical equilibrium (under the Guldberg-Waage law of mass action) is treated in an abstract way with elementary methods of the theory of partially ordered linear spaces (vector lattices). Systems formed from n given components are represented as points $x>0$ of such a space K (of n dimensions); reactions are represented as $x=y$, $x\in K$, $y\in K$, $x>0$, $y>0$. Given r reactions $x^i=y^i$, let L denote the linear subspace spanned by all x^i-y^i. For any $p\in K$, $p=\sum\pi_ie_i$, $\pi_i>0$ (e_i are basic vectors), let $\psi(p)$ be the linear form on K assuming values $\log\pi_i-\log\sum_{i=1}^m\pi_i$ at $e_i\in K'$, and 0 at $e_i\in K''$, where K', spanned by basic vectors $e_1, \cdots, e_m$, is the subspace representing systems containing gaseous components only, and K'' is the complementary subspace. The well-known equilibrium condition may be formulated as $\psi(p)(x)=f(x)$ for all $x\in L$; here, f is a linear form on K representing (to a multiplicative constant) the free energy. Several results, mainly concerning ψ, are obtained and illustrated with two examples. Some problems are stated; e.g., under what conditions, for a given positive $a\in K$, is the set of all $\psi(p)$, $p-a\in L$, linear.

{Reviewer's remarks: It is easy to show that the transformation ψ is open, from which (and from Theorem 2) Theorem 3 follows at once. The assertion (p. 456) that $\psi(O_a)$ is open in D is clearly an error: $\psi(O_a)$ is open in a suitable linear subspace of D.} *M. Katětov* (Prague)

2601:

Hotta, Jyôji. On Yamamuro's theorem concerned with linear modulars. Proc. Japan Acad. 33 (1957), 600–602.

The author gives a simple, natural proof of the theorem of S. Yamamuro [J. Fac. Sci. Hokkaido Univ. Ser. I **12** (1953), 211–253; MR **16**, 50] which says: If the first and second norms of a modulared semi-ordered linear space coincide with each other, then the modular is either linear or singular. *I. G. Amemiya* (Kingston, Ont.)

2602:

Hirschfeld, Rudi. Sur la théorie générale des meilleures approximations. C. R. Acad. Sci. Paris **246** (1958), 1485–1488.

Let E be a normed vector space, $x \in E$, G a closed vector subspace of E. Consider the existence and uniqueness of elements $y \in G$ such that

$$\|x-y\|=\operatorname{Inf}_{g\in G}\|x-g\|.$$

Although y, if it exists, may fail to be unique, the author writes $y=A_G x$ and calls A_G the "approximation operator". E is said to possess property (E) (resp. (U), resp. (L)) if $A_G x$ exists for each G and a dense set of x (resp. $A_G x$ is unique whenever it exists, resp. A_G is linear). Several known results and some existing literature are recalled rapidly, and some new theorems are added. Thus (Théorème 1) each reflexive space E enjoys property (E); again, completing a theorem of M. G. Krein, one has (Théorème 3) the criterion that E possesses (U) if and only if it is strictly convex. Théorème 2 asserts that the strong dual of a normed space enjoys (E) relative to its weakly closed vector subspaces G. Théorème 4 gives a characterisation of prehilbert spaces in terms of (E) and (U). Sketch proofs only are given.

{Reviewer's note: In the first line of the sketch proof of Théorème 1, "compacte" should read "faiblement compacte".} *R. E. Edwards* (Woking)

2603:

Singer, Ivan. Sur le L-problème de la théorie des moments dans les espaces de Banach. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 19–28. (Romanian. Russian and French summaries)

Suppose that G is a linear subspace of a normed space E and that F is a continuous linear functional on G of norm one. By the Hahn-Banach theorem f admits an extension F to all of E such that $F \in S^*$, the unit sphere of E^*. The author proves that if G is n-dimensional, then F can be chosen to be a convex combination of h extreme points of S^*, $1 \leq h \leq n$. This is applied to yield new proofs of known theorems due to the author and others.

R. R. Phelps (Princeton, N.J.)

2604:

Singer, Ivan. Sur l'unicité de l'élément de meilleure approximation dans des espaces de Banach quelconques. Acad. R. P. Romîne. Stud. Cerc. Mat. **8** (1957), 235–244. (Romanian. Russian and French summaries)

Lengthy and detailed proofs of theorems presented previously by the author [Acta Sci. Math. Szeged, **17** (1956), 181–189; MR **18**, 891; Theorem 3.1; Acad. R. P. Romîne. Stud. Cerc. Mat. **7** (1956), 95–145; MR **18**, 221; Theorem 2.2], plus a correction to Theorem 2.1 of the latter paper. *R. R. Phelps* (Princeton, N.J.)

2605:

Singer, Ivan. Angles abstraits et fonctions trigonométriques dans les espaces de Banach. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 29–42. (Romanian. Russian and French summaries)

For nonzero x and y in a Banach space E, the author defines $\sin_E \frac{1}{2}\langle x, y\rangle=\frac{1}{2}\|(x/\|x\|-y/\|y\|)\|$, $\cos_E \frac{1}{2}\langle x, y\rangle=\frac{1}{2}\|(x/\|x\|+y/\|y\|)\|$, where $\langle x, y\rangle$ is an "abstract angle". The sign of $\cos_E \frac{1}{2}\langle x, y\rangle$ is ambiguous, being determined by the choice of a planar representation of the span of x and y (if E is the Euclidean plane, then $\langle x, y\rangle$ is the smallest positive angle with x and y as initial and terminal sides). It follows from a theorem of M. M. Day [Trans. Amer. Math. Soc. **62** (1947), 320-337; MR **9**, 192] that $\sin_E^2 \frac{1}{2}\langle x, y\rangle+\cos_E^2 \frac{1}{2}\langle x, y\rangle=1$ for all nonzero x and y if and only if E is a Hilbert space. The author defines $x \perp y$ to mean $|\tan_E \frac{1}{2}\langle x, y\rangle|=1$, i.e., $\|(x/\|x\|+y/\|y\|)\|=\|(x/\|x\|-y/\|y\|)\|$. This is compared to the orthogonality ($\|x+\lambda y\|=\|x-\lambda y\|$ for all λ) of B. D. Roberts [Tôhoku Math. J. **39** (1934), 42–59] and "isosceles orthogonality" ($\|x+y\|=\|x-y\|$) of R. C. James [Duke Math. J. **12** (1945), 292–302; MR **6**, 273]. For any linearly independent x and y, there is a number a for which $x \perp ax+y$ (also true for "isosceles orthogonality"). If $x \perp y$, then $ax \perp by$ for a and b not both zero (also true for Roberts' orthogonality).

R. C. James (Claremont, Calif.)

2606:

Iséki, Kiyoshi. On complete orthonormal sets in Hilbert space. Proc. Japan Acad. 33 (1957), 450–452.

There are many versions of the principle which states that a set sufficiently close to a complete orthonormal set in Hilbert space is itself complete. Let $\{\varphi_n\}$ be a complete orthonormal set, $r_n=\|\varphi_n-\psi_n\|$. Hilding has shown that $\{\psi_n\}$ is also complete if (i) $\|\psi_n\|=1$ and $\sum_{n=1}^{\infty} r_n^2(1-r_n^2/4)<1$, or (ii) $(\varphi_n, \psi_n)=1$ and $\sum_{n=1}^{\infty} r_n^2/(1+r_n^2)<1$ [Ark. Mat. Astr. Fys. **32B**, no. 7, 3 pp. (1946); MR **8**, 151]. The present author shows that if $\{\psi_n\}$ is orthogonal mere convergence of the series in Hilding's conditions (i) and (ii) (instead of convergence to a sum <1) will suffice. {Reviewer's remark: The author's result may be derived from Hilding's in the following way. Let $\sum_{n>N}<1$. Then the set $\varphi_1, \cdots, \varphi_N, \psi_{N+1}, \psi_{N+2}, \cdots$ is complete. Thus the subspace orthogonal to the vectors $\psi_{N+1}, \psi_{N+2}, \cdots$ is at most N-dimensional, and therefore generated by $\psi_1, \cdots, \psi_N$.} *J. Korevaar* (Madison, Wis.)

2607:

Kaz'min, Yu. A. On bases and complete systems of functions in Hilbert space. Mat. Sb. N.S. **42(84)** (1957), 513–522. (Russian)

Proofs of theorems announced in Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 1199–1202 [MR **19**, 852].

M. Jerison (Princeton, N.J.)

2608:

Mackey, George W. Quantum mechanics and Hilbert space. Amer. Math. Monthly **64** (1957), no. 8, part II, 45–57.

A stimulating exposition, with several points of novelty, of the mathematical formulation of Schroedinger theory. An observable taking values 0 or 1 with probability 1 ("Eigenschaft" in von Neumann's book [Mathematische Grundlagen der Quantenmechanik, Springer, Berlin, 1932]) is here called a "question". States on the observables are determined by, and may be discussed in terms of, their values on the questions. The author assumes that the questions may be identified with the lattice of all closed subspaces M of a Hilbert space $\mathfrak{H}$; and that a measure on the questions corresponds to a state only if it is a convex combination of those obtained from $\phi \in \mathfrak{H}$ by $M \to (P_M\phi, \phi)$. He asks whether other measures exist [see

#2609 below]. There is discussion of the role of the Hamiltonian, but not of any specific examples.

C. Davis (Providence, R.I.)

2609:

Gleason, Andrew M. Measures on the closed subspaces of a Hilbert space. J. Math. Mech. **6** (1957), 885–893.

The author establishes a conjecture of Mackey [#2608 above] to the effect that a completely-additive non-negative real-valued function m on the closed linear subspaces of a Hilbert space of dimension greater than two has the form $m(K)=\text{trace}(DP_k)$, where P_k denotes the projection on the space with range K, and D is a fixed operator of trace class. The proof reduces readily to the treatment of the three-dimensional case, which involves notably the demonstration of the continuity as a function of the vector x of $m(P(x))$, where $P(x)$ is the projection whose range is the one-dimensional subspace spanned by x. To what extent a similar conclusion holds in general C^*-algebras — e.g., for which algebras every positive homogeneous functional that is completely additive on commuting elements is additive generally, a question relevant both to the problem treated by Mackey of defining physically the sum of observables and to the additivity of the trace in algebras of finite type — is left open, as the author's method is applicable only to rings of operators of type I. The conclusion fails in the two-dimensional case, which so far as is known may be essentially the only such case.

I. E. Segal (Copenhagen)

2610:

Zuhovickiĭ, S. I.; and Èskin, G. I. The approximation of abstract continuous functions by unbounded operator-functions. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 731–734. (Russian)

This note announces generalizations of earlier results of Zuhovickiĭ and Stečkin [Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 385–388; Mat. Sb. N.S. **37(79)** (1955), 3–20; MR **18**, 222; **17**, 388]. Let H_i be Hilbert spaces with zeros θ_i and norms $\|\ \|_i$ $(i=1, 2)$. Let Q be a compact metric space and A a mapping of Q into the set of closed linear operators with domains $\subset H_1$ and ranges $\subset H_2$ such that for every $q\in Q$, the domain of $A(q)$ is a fixed dense subspace D of H_1. Suppose also that for every $x\in D$, the function $A(q)x$ mapping Q into H_2 is continuous. Suppose finally that $x\in D$ and $A(q)x=\theta_2$ for all $q\in Q$ implies that $x=\theta_1$. For a continuous mapping f of Q into H_2, one may look for an $x_0\in D$ such that

$$\max_{q\in Q}\|A(q)x_0-f(q)\|_2=\inf_{x\in D}\{\max_{q\in Q}\|A(q)x-f(q)\|_2\}.$$

Such an x_0 exists for every such f if and only if there is a real number $m>0$ such that $\max_{q\in Q}\|A(q)x\|_2\geqq m\|x\|_1$ for all $x\in D$. Various applications and generalizations of this result are pointed out. No proofs are given.

E. Hewitt (Seattle, Wash.)

2611:

Wada, Junzo. Stonian spaces and the second conjugate spaces of AM spaces. Osaka Math. J. **9** (1957), 195–200.

"We consider an AM space $C(X)$ which is the second conjugate space of an AM space. Such a space X is hyperstonian, and if the character of X is countable, then X is the space βN_0, where N_0 is a discrete space whose cardinal number is at most countable." {Hyperstonian: compact extremally disconnected, and the union of the supports of all measure which vanish on all non-dense sets is dense. Countable character: no disjoint uncountable family of open sets.}

J. Isbell (Seattle, Wash.)

2612:

Iséki, Kiyoshi. On a theorem on function space of A. Grothendieck. Proc. Japan Acad. **33** (1957), 605–607.

A result on conditional compactness of subsets A of the space $C_s(E)$ of real continuous functions on E under the topology of simple convergence, proved by A. Grothendieck [Amer. J. Math. **74** (1952), 168–186; MR **13**, 857] for E countably compact, is extended to E pseudo-compact (=every real continuous function is bounded). It is seen that the following properties for A are equivalent: (1) A is conditionally compact; (2) A is conditionally countably compact; (3) A is pointwise bounded and for any sequence (f_n) in A and (x_n) in E, the double sequence $(f_m(x_n))$ has at least one double cluster point (=a point such that, for each of its neighborhoods U and for each integer N, there are infinitely many $f_m(x_n)$ in U with $m, n\geqq N$). In addition, a condition like (3) implies that E is pseudo-compact.

R. G. Bartle (Urbana, Ill.)

2613:

Holladay, John C. On the identity of function spaces on Cartesian product spaces. Proc. Amer. Math. Soc. **9** (1958), 44–46.

For $i=1, 2, \cdots, n$, let F_i be a closed linear subspace of the Banach space $C(S_i)$ of all continuous (real or complex) functions on a compact space S_i. Let $F_1\otimes\cdots\otimes F_n$ denote the closure in $C(S_1\times\cdots\times S_n)$ of the tensor product of $F_1, \cdots, F_n$, and let $F_1*\cdots*F_n$ consist of those functions in $C(S_1\times\cdots\times S_n)$ that yield members of F_i when all coordinates but the ith are fixed. As a contribution to the question of whether these products are identical, the author proves that if the (nonzero) subspaces F_i and G_i differ by a finite number of dimensions, then $F_1\otimes\cdots\otimes F_n=F_1*\cdots*F_n$ if and only if $G_1\otimes\cdots\otimes G_n=G_1*\cdots*G_n$.

M. Jerison (Princeton, N.J.)

2614:

Ishii, Tadashi. On homomorphisms of the ring of continuous functions onto the real numbers. Proc. Japan Acad. **33** (1957), 419–423.

Let $\mathfrak{C}_X$ be any subring of the ring $C(X, R)$ of all real-valued continuous functions on a completely regular space X that satisfies the conditions: (1) $R\subset\mathfrak{C}_X$; (2) for a closed set F of X and a point $p\notin F$, there exists $f\in\mathfrak{C}_X$ such that $f(p)>\sup_{x\in F}f(x)$; (3) if $f(x)>a>0$ for $x\in X$, and $f\in\mathfrak{C}_X$, then $f^{-1}\in\mathfrak{C}_X$. Let gX denote the uniform space X with the smallest uniform structure in which every function in $\mathfrak{C}_X$ is uniformly continuous. Theorem 1: For every homomorphism ϕ of the ring $\mathfrak{C}_X$ onto R, there exists a unique $p\in X$ such that $\phi(f)=f(p)$ for all $f\in\mathfrak{C}_X$, if and only if gX is complete. From this is proved Theorem 2: If the uniform spaces gX and gY determined by $\mathfrak{C}_X$ and $\mathfrak{C}_Y$ are complete, and the rings $\mathfrak{C}_X$ and $\mathfrak{C}_Y$ are isomorphic, then X and Y are homeomorphic. Moreover, if η is the homeomorphism from X to Y, then $f\eta^{-1}\in\mathfrak{C}_Y$ for all $f\in\mathfrak{C}_X$ and $f'\eta\in\mathfrak{C}_X$ for all $f'\in\mathfrak{C}_Y$. The special case $\mathfrak{C}_X=C(X, R)$, $\mathfrak{C}_Y=C(Y, R)$ is Th. 57 of Hewitt, Trans. Amer. Math. Soc. **64** (1948), 45–99 [MR **10**, 126].

The author shows next that if X is a locally compact, σ-compact Hausdorff space, and $\mathfrak{C}$ is a subring of $C(X, R)$ that contains the constants, completely separates points and closed sets (i.e., $p\notin F$ implies $f(F)=0$, $f(p)=1$ for some $f\in\mathfrak{C}$), and such that $\sum_n f_n\in\mathfrak{C}$ whenever $\{f_n\}$ is a sequence of nonnegative functions in $\mathfrak{C}$ with $\{P(f_n)\}$ locally finite (where $P(f_n)=\{x\in X: f_n(x)>0\}$), then the uniform space gX determined by $\mathfrak{C}$ is complete. By combining this with Theorem 1, one obtains the (known) corollary that the ring of all C^∞-functions on a (separable)

C^∞-manifold X characterizes the C^∞-structure of X.

Another result, proved by a different argument, is that if X is a locally compact Hausdorff space and ϕ is a homomorphism onto R of the subring $\mathfrak{C}_k$ of $C(X, R)$ of functions with compact supports, then there exists a unique $p \in X$ such that $\phi(f)=f(p)$ for all $f \in \mathfrak{C}_k$. A corollary is that $\mathfrak{C}_k$ characterizes the topology of the locally compact Hausdorff space X; this was reported earlier by Shanks [Bull. Amer. Math. Soc. **57** (1951), 295].

C. W. Kohls (Urbana, Ill.)

2615:

Zerner, Martin. Sur la régularité de certaines familles de distributions. C. R. Acad. Sci. Paris **246** (1958), 686–687.

L'auteur démontre le théorème suivant: toute distribution en (x, y), $x \in R^m$, $y \in R^n$, qui soit semi-régulière en x et intégralement semi-régulière en y, est une fonction indéfiniment dérivable de (x, y).

J. Sebastião e Silva (Lisbonne)

2616:

Lojasiewicz, S. Division d'une distribution par une fonction analytique de variables réelles. C. R. Acad. Sci. Paris **246** (1958), 683–686.

The author gives an affirmative answer to the following conjecture of L. Schwartz: Let F be a real analytic function and T a distribution; there exists a distribution S satisfying $FS=T$ (problem of division). A special case of this result was proven for two dimensions and for F a polynomial by McKibben [Amer. J. Math. **81** (1959), 23–36]. The main point in the proof is the following result which is stated without proof: Let $\Phi(x_1, \cdots, x_k)$ be a real function of real variables which is analytic in an open neighborhood of a compact set K. Let $\rho(x_1, \cdots, x_k)$ denote the distance from K to the set where $\Phi=0$ (supposed non-empty). Then there exist positive constants c, q so that $|\Phi| \geqq c\rho^q$ for $x \in K$.

L. Ehrenpreis (Waltham, Mass.)

2617:

Lojasiewicz, S. Sur l'identification d'une classe de fonctions (non nécessairement sommables) avec des distributions. C. R. Acad. Sci. Paris **246** (1958), 872–874.

The distribution T on the real line is said by the author to be regular at a point c if its primitive has a value at c [Studia Math. **16** (1957), 1–36; MR **19**, 433]. Let U be the class of distributions regular at each point of the line. For each $T \in U$ and for each interval Δ there exists a unique $T_\Delta \in U$ such that $T_\Delta=T$ in the interior of Δ and $T_\Delta=0$ in the exterior of Δ. Let F be a closed set which is a union of disjoint closed intervals Δ_i. F is called regular with respect to T if the series $\sum T_{\Delta_i}$ converges to a distributions in U. We set $T_F=T-\sum T_{\Delta_i}$, so $T_F \in U$.

Let $\mathfrak{T}$ be a subclass of U; we define $\mathfrak{T}^* \subset U$ to consist of all $T \in U$ for which there exists a set F which is regular with respect to T such that $T_F \in \mathfrak{T}$ and $T_\Delta \in \mathfrak{T}$ if Δ is a finite interval contained in the complement of F.

Now, start with the class $\mathfrak{L}$ of distributions which are locally summable functions and define $\{\mathfrak{L}_\xi\}$ by: $\mathfrak{L}_0=\mathfrak{L}$, and $\mathfrak{L}_\xi=(\bigcup_{\eta<\xi} \mathfrak{L}_\eta)^*$; call $\mathfrak{F}=\bigcup \mathfrak{L}_\xi$. The author characterizes the set $\mathfrak{F}$ and shows that it contains the set of distributions which have a value except for a denumerable set.

Let F be regular with respect to $T \in U$. Suppose x_0 is a point of condensation of F and T_F is a summable function whose restriction f to F is continuous at x_0. Then we call $f(x_0)$ the approximate value of T at x_0 and we write $T_a(x_0)=f(x_0)$ .It may happen that T has both a value at x_0 and an approximate value at x_0 which are different. However, we have: If $T \in U$ and $T(x)$ and $T_a(x)$ exist everywhere on a measurable set E then $T(x)=T_a(x)$ a.e. on E. Moreover, each $T \in \mathfrak{F}$ possesses an approximate value a.e.

Finally, the author states that the class $\mathfrak{F}$ contains all distributions which correspond (in the usual way) to functions which are Denjoy locally integrable.

No proofs are given. *L. Ehrenpreis* (Waltham, Mass.)

2618:

Sato, Mikio. On a generalization of the concept of functions. Proc. Japan Acad. **34** (1958), 126–130.

The author generalizes the theory of distributions of L. Schwartz by using the natural embedding of real Euclidean space in the complex Euclidean space. The approach is similar to that of Köthe [Math. Z. **57** (1952), 13–33; MR **14**, 563]. *L. Ehrenpreis* (Waltham, Mass.)

2619a:

Roumieu, Charles. Une extension de la notion de distribution. C. R. Acad. Sci. Paris **246** (1958), 520–521.

2619b:

Roumieu, Charles. Transformation de Fourier, des distributions généralisées. Applications. C. R. Acad. Sci. Paris **246** (1958), 678–680.

L'A. remplace les espaces de fonctions indéfiniment differentiables de L. Schwartz, en dimension 1 [Théorie des distributions, t. I, II, Hermann, Paris, 1950, 1951; MR **12**, 31, 833], par des espaces de fonctions indéfiniment differentiables à support compact vérifiant des conditions supplémentaires du type $|f^{(\nu)}(t)| \leqq Ak^\nu M_\nu$, $\nu=0, 1, \cdots$, $M_\nu>0$, $M_\nu^2 \leqq M_{\nu-1}M_{\nu+1}$, $\sum M_\nu^{-1/\nu}<\infty$.

Muni d'une topologie convenable, le dual de cet espace est appelé espace des distributions généralisées (il contient l'espace des distributions de L. Schwartz).

Notion de support; introduction de l'espace des distributions généralisées à support compact.

Moyennant la condition supplémentaire

$$\limsup \nu^{-2} \log M_\nu<\infty,$$

toute dérivée de distribution généralisée est une distribution généralisée.

Généralisation de l'espace S de Schwartz (fonctions indéfiniment différentiables à décroissance rapide) — Définition d'une transformé de Fourier généralisée.

Ces notions introduites, l'A. annonce trois résultats intéressants: 1) une généralisation du théorème des supports de produit de convolution; 2) un théorème de structure remarquable; 3) une condition nécessaire et suffisante de résolution d'une équation de convolution (tout ceci en dimension 1; il serait intéressant de généraliser cela en dimension n).

{L'A. ne semble pas au courant des travaux déjà publiés dans cette direction: Gelfand-Silov, Hörmander, Silov, etc....; voir notamment une bibliographie abondante dans Gelfand et Silov, J. Math. Pures Appl. (9) **35** (1956), 383–413 [MR **18**, 493]; mais néanmoins les résultats essentiels de l'A. sont, à la connaissance du rapporteur, nouveaux.} *J. L. Lions* (Nancy)

2620:

Hukuhara, Masuo; et Sibuya, Yasutaka. Théorie des endomorphismes complètement continus. II. J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1958), 511–525.

[For first part see same J. **7** (1957), 391–405; MR **18**, 909.] The authors continue their detailed presentation of the theory of completely continuous (c.c.) linear operators

in a linear ($\mathfrak{L}$)-space, as indicated in a previons note [Proc. Japan. Acad. **31** (1955), 595–599; MR **17**, 645]. Iu the present article they treat the proper values of a c.c. operator, and the series development of the resolvent. A decomposition formula is obtained for c.c. endomorphisms of a dual pair of linear spaces. No references are supplied.

J. H. Williamson (Cambridge, England)

2621:

Ringrose, J. R. Complete continuity conditions on linear operators. Proc. London Math. Soc. (3) **8** (1958), 343–356.

Let A be a continuous linear transformation of a topological vector space E into another such space F. Natural definitions are given for A to be completely continuous (c.c.), locally c.c., boundedly c.c., precompact, boundedly precompact and compact. For example, A is precompact if there is a neighborhood V in E such that $A(V)$ is a precompact subset of F; A is c.c. if it is continuous as a mapping from E in its weak topology into F. Interesting inter-relations and properties of these notions are derived. The two notions of precompactness and compactness coincide when E and F are Banach spaces, but in general do not. Examples given show that some of the connections known for these ideas for Banach spaces do not extend to the locally convex case. The adjoint operator A' is investigated. It is shown that (for $E=F$) if A is precompact then $(A')^2$ is compact.

B. Yood (New Haven, Conn.)

2622:

Donoghue, William F., Jr. On the numerical range of a bounded operator. Michigan Math. J. **4** (1957), 261–263.

The numerical range $W(T)$ of the bounded operator T on a complex Hilbert space, is the set of all (Wf, f) $(\|f\|=1)$. The principal lemma is a complete description (only part of it new) of $W(T)$ in the 2-dimensional case; the proof is short and elementary. Now in higher-dimensional cases, either of the properties (i) T is a multiple of the identity, (ii) T is self-adjoint, will hold for T if and only if it holds for each ETE (E a 2-dimensional projection). Thus the author deduces immediately from his lemma the known characterizations, in terms of $W(T)$, of operators with property (i) or (ii) respectively. Similarly the lemma yields, via the ETE, Hausdorff's theorem that $W(T)$ is convex, and two new theorems concerning boundary points of $W(T)$. The facts here differ from those found by Meng [Proc. Amer. Math. Soc. **8** (1957), 85–88; MR **20** #1223] in case T is required to be normal.

C. Davis (Providence, R.I.)

2623:

Schaefer, Helmut. Über singuläre Integralgleichungen und eine Klasse von Homomorphismen in lokalkonvexen Räumen. Math. Z. **66** (1956), 147-163.

Let X be a Hilbert space and R the ring of linear continuous operators from X into X. A linear continuous operator T from X into X is called a generalized Fredholm operator or a σ-transformation if the following conditions are satisfied: 1) $T(X)$ is closed and the factor space $X/T(X)$ has finite dimension $n(T)$; 2) the null-subspace of T has finite dimension $m(T)$. The set Σ of all σ-transformations forms a semigroup such that, if $T\in\Sigma$, then also $T^*\in\Sigma$, where T^* is the adjoint of T. It is shown that, if $T\in R$ and dim $X\geqq N_0$, then $T\in\Sigma$ if and only if there exist $U, V\in\Sigma$ such that $UTV=I$, where I is the identity transformation. This theorem is used in the proof of the following well-known properties of σ-transformations. (a) Σ is an open subset of a Banach algebra R. (b) If $T\in\Sigma$, then $T+K\in\Sigma$ for every linear completely continuous transformation K of X into X. (c) If $T_1, T_2\in\Sigma$, then $\gamma(T_1T_2)=\gamma(T_1)+\gamma(T_2)$, where $\gamma(T)$ is the index of the transformation T; that is, $\gamma(T)=m(T)-n(T)$. (d) Let $T_0\in\Sigma$ be fixed, let A be a linear transformation satisfying the condition: $\|A\|<r$ with suitable positive number r, and let K be an arbitrary linear, completely continuous operator. Then $\gamma(T_0+A)=\gamma(T_0+K)=\gamma(T_0)$. (e) Let $T_0\in\Sigma$ and K be a completely continuous operator. Then, with the exception of at most countably many points $\{\lambda_v\}$ without limit points in the complex λ-plane, $m(T_0-\lambda K)=m_0$ and $n(T_0+\lambda K)=n_0$ are independent of λ, and at the points λ_v (the set of which may be empty) $m(T_0+\lambda_v K)>m_0$, $n(T_0+\lambda_v K)>n_0$. These theorems are used in the proof of fundamental properties of singular integral equations in the case of one dimension.

In his final paragraph the author considers homomorphisms of a locally convex linear topological Hausdorff space $\mathfrak{E}$ with another such space $\mathfrak{F}$ and, of these, distinguishes the set $\Sigma(\mathfrak{E},\mathfrak{F})$ of all σ-transformations defined as before. The following theorems are proved. (1) If T is a linear continuous operator from $\mathfrak{E}$ to $\mathfrak{F}$, then the condition that the equation $Tx=y$ have normalized solutions is equivalent to the condition that $T(\mathfrak{E})$ be closed. (2) $T\in\Sigma(\mathfrak{E},\mathfrak{F})$ if and only if there exist two linear continuous transformations U and V such that $UT=I-L_1$, $TV=I-L_2$, where I is the identity transformation and L_1, L_2 are linear continuous transformations of finite rank from $\mathfrak{E}$ into $\mathfrak{E}$ and from $\mathfrak{F}$ into $\mathfrak{F}$, respectively. (3) If $T\in\Sigma(\mathfrak{F},\mathfrak{G})$, $S\in\Sigma(\mathfrak{E},\mathfrak{F})$, then $TS\in\Sigma(\mathfrak{E},\mathfrak{G})$, whith $\gamma(TS)=\gamma(T)+\gamma(S)$. (4) Let S and T be linear continuous transformations from $\mathfrak{E}$ into $\mathfrak{F}$ and from $\mathfrak{F}$ into $\mathfrak{G}$ respectively. Then, if $TS\in\Sigma(\mathfrak{E},\mathfrak{G})$, we have either $T\notin\Sigma(\mathfrak{F},\mathfrak{G})$ and $S\notin\Sigma(\mathfrak{E},\mathfrak{F})$, or $T\in\Sigma(\mathfrak{F},\mathfrak{G})$ and $S\in\Sigma(\mathfrak{E},\mathfrak{F})$. (5) Let $T_0\in\Sigma(\mathfrak{E},\mathfrak{F})$ and K be an arbitrary linear completely continuous transformation from $\mathfrak{E}$ into $\mathfrak{F}$. Then $T_0+K\in\Sigma(\mathfrak{E},\mathfrak{F})$ and $\gamma(T_0+K)=\gamma(T_0)$. There is a bibliography with 23 entries.

M. M. Vaĭnberg (RŽMat **1957** #7941)

2624:

Nakamura, Masahiro. On operators of Schaefer class in the theory of singular integral equations. Proc. Japan Acad. **33** (1957), 455–456.

A bounded linear operator is said to belong to the Schaefer class if its range is closed and of finite codimension and its null-space (or kernel) is of finite dimension. The author shows that a bounded linear operator on a separable Hilbert space belongs to the Schaefer class if and only if it corresponds to a regular element of the quotient algebra of the bounded linear operators modulo the compact linear operators. This leads to very short proofs of some of Helmut Schaefer's results [#2623 above].

{The principal results of this paper can be extended to a general Banach space.}

A. F. Ruston (Sheffield)

2625:

Kato, Tosio. Perturbation of a scattering operator and its continuous spectrum. Sûgaku **9** (1957/58), 75–84. (Japanese)

Beginning with a historical survey of the connection between the "scattering operators (=S matrices)" and the perturbation of continuous spectrum of a self-adjoint operator in a Hilbert space X, as developed by K. Friedrichs [Comm. Pure Appl. Math. **1** (1948), 361–406; MR **10**, 547], M. Rosenbloom [Bull. Amer. Math. Soc. **62** (1956), 30] and N. Aronszajn [ibid. **63** (1957), 105], the

author gives an excellent exposition, with detailed proof, of his investigations [Proc. Japan. Acad. **33** (1957), 260–264; J. Math. Soc. Japan **9** (1957), 239–249; MR **19**, 1068].

A bounded self-adjoint operator B in X is said, following after R. Schatten, to belong to the "trace family" T_s if $\|B\|_1=\|(B^*B)^{\frac{1}{2}}\|_2{}^2<\infty$ where $\|B\|_2=(\sum_\lambda \|B\varphi_\lambda\|^2)^{\frac{1}{2}}$, $\{\varphi_\lambda\}$ denoting a complete orthonormal system of X. For a self-adjoint operator $H=\int \lambda dE(\lambda)$ in X, define the absolutely continuous part $A(H)$, of X with respect to H, as the subspace of X spanned by the totality of $x\in X$ such that the measure $(E(\lambda)x, x)$ on $-\infty<\lambda<\infty$ is absolutely continuous with respect to the Lebesgue measure on $-\infty<\lambda<\infty$. Th. 1: If H_0 is self-adjoint and $V\in T_s$, then $H_1=H_0+V$ is self-adjoint with the same domain as H_0, and the strong-$\lim_{t\to\pm\infty} \exp(itH_1)\exp(-itH_0)\operatorname{Proj}(A(H_0)) =U_\pm$ exist as partially isometric operators from $A(H_0)$ onto $A(H_1)$. Moreover, H_0 and H_1, if restricted respectively upon $A(H_0)$ and $A(H_1)$, are unitarily equivalent. The scattering operator $S=U_+{}^*U_-$ is unitary in $A(H_0)$.

The proof of these results is reduced, by a series of propositions, to the case where the "perturbation term" V is of finite rank.

K. Yosida (Tokyo)

2626:

Wenjen, Chien. On semi-normed *-algebras. Pacific J. Math. **8** (1958), 177–186.

The author presents a relation of the spectral theorem for unbounded self-adjoint operators to certain convex topological *-algebras. This relation is an analogue of the well-known relation of the spectral theorem for bounded self-adjoint operators to Banach *-algebras. He does not pretend to give a substantially new proof of the spectral theorem, inasmuch as the available tool for studying the *-algebras in question [R. Arens, same J. **2** (1952), 455–471; MR **14**, 482; E. Michael, Mem. Amer. Math. Soc. no. 11 (1952); MR **14**, 482], is systematic homomorphism into algebras. A variety of generalizations of the lemma and theorem of I. Gel'fand and M. Neumark [Mat. Sb. N.S. **12** (1943), 197–213; MR **5**, 147] are also obtained.

R. Arens (Los Angeles, Calif.)

2627:

Ghika, Al. Décompositions spectrales généralisées des transformations linéaires d'un espace hilbertien dans un autre. Rev. Math. Pures Appl. **2** (1957), 61–109.

Let H be a Hilbert space with elements $x, x', \cdots$ and scalar product $(x|x')$. The usual spectral theory refers to a certain class of linear operators T with domain H_T dense in H, and range in H. It involves the notions of a spectral family of projections belonging to T, and (for bounded T) the notion of the subalgebra generated by T, of the algebra $\mathscr{L}(H)$ of bounded linear operators of H into itself.

The present paper is concerned with the case where the range of T is in a second Hilbert space $\mathscr{J}$ with elements $y, y', \cdots$ and scalar product $(y|y')$. This generalization uses in an essential way the notion of a partially isometric mapping W of H into $\mathscr{J}$, i.e., of a mapping which maps some closed linear submanifold F of H isometrically into $\mathscr{J}$, and maps each point of the orthogonal complement $H\ominus F$ into the zero element of $\mathscr{J}$. The adjoint W^* is then a partially isometric map of $\mathscr{J}$ into H. Particularly, use is made of the following fact proved by Murray [Trans. Amer. Math. Soc. **37** (1935), 301–338]: A linear operator T densely defined in H with range in $\mathscr{J}$ allows the decomposition $R=W(T^*T)^{\frac{1}{2}}$ with partially isometric W.

Such a fixed partially isometric W is used to make the Banach space $\mathscr{L}(H, \mathscr{J})$ of continuous linear maps $H\to\mathscr{J}$ into a Banach algebra $\mathscr{L}_{W^*}(H, \mathscr{J})$ by defining multiplication as follows: (1) $A\circ B=AW^*B$, A and B in $\mathscr{L}(H,\mathscr{J})$. W is an idempotent of norm 1 in this algebra. For fixed T in $\mathscr{L}(H, \mathscr{J})$ the subalgebra $\mathfrak{A}_{W^*}(T)$ of $\mathscr{L}_{W^*}(H, \mathscr{J})$ generated by W and T is then the closure of all elements of the form $\alpha_0 W+\sum_{i=1}^n \alpha_i T^i$ where the α_i are complex numbers and the powers T^i are defined in terms of the multiplication rule (1). The algebra $\mathfrak{A}_{W^*}(T)$ is shown to be isomorphic to the subalgebra $\mathfrak{A}(T_T)$ of the Banach algebra $\mathscr{L}(H)$ generated by $T_T=(T^*T)^{\frac{1}{2}}$ under the map $A\to W^*A$.

An element Q of $\mathscr{L}(H, \mathscr{J})$ is called a W-projection if $Q=WW^*Q=QW^*W$ and W^*Q is a projection in H. With this definition of a W-projection and the definition (1) of multiplication the notion of a W-spectral family of projections Q_λ depending on a real parameter λ is then defined by the usual properties except that Q_λ is defined only for nonnegative λ and that therefore the usual condition for $\lambda\to-\infty$ is replaced by $Q_0=0$; moreover, the usual condition for $\lambda\to+\infty$ is replaced by requiring $\lim_{\lambda\to+\infty} Q_\lambda=W$.

Let now T be a closed linear map densely defined in H with range in $\mathscr{J}$. Then by Murray's theorem mentioned above there exists one and only one partially isometric W such that T is positive and "self-adjoint with respect to W", i.e., that $T=WW^*T=TW^*W$ and that the operator $B=W^*T$ (with domain and range in H) is self adjoint in the usual sense. It is then proved that with this W there exists one and only one W-spectral family W_λ such that

$$(Tx|y)=\int_0^\infty \lambda d(W_\lambda x|y)$$

for x in the domain of T and y in $\mathscr{J}$.

If in particular T is continuous then the spectrum of T lies on the interval $[0, \|T\|]$ and the algebra $\mathfrak{A}_{W^*}(T)$ is isomorphic to the algebra of continuous complex valued functions defined on the spectrum.

E. H. Rothe (Ann Arbor, Mich.)

2628:

Ionescu Tulcea, Cassius. Spectral representation of semigroups of normal operators. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 44–45.

Let G be a locally compact group with Haar measure μ; S a subset of G with: (1) $xy=yx\in S$ for all x, y in S; (2) $\mu(U)>0$ for all non-void $U\subset S$ open in S. Let E be the space of characters on S endowed with the topology of uniform convergence on compact sets. If $\mathscr{U}$ is a class of relatively open and compact sets in S which covers S and $R=\{r(X)\}_{X\in\mathscr{U}}$ is a set of real numbers not less than one, let $E(R)$ be the subspace of characters $\chi\in E$ with $|\chi(t)|\leq [r(X)]^{\frac{1}{2}}$ for $t\in X$, $X\in\mathscr{U}$. Let H be a Hilbert space and let $\varphi(t)=U_t$ be a weakly continuous map of S into the set of bounded normal operators on H such that $U_tU_s=U_{ts}$ for $t, s\in S$ and $\|U_t\|\leq[r(X)]^{\frac{1}{2}}$ for $t\in X$, $X\in\mathscr{U}$.

The following theorem is stated. There exists a unique spectral family of measures $(\mu_{x,y})_{x,y\in H}$ on $E(R)$ such that for $t\in S$, $x, y\in H$, we have (α) $(U_tx|y)=\int_{E(R)} \chi(t)d\mu_{x,y}(\chi)$; ($\beta$) $\|\mu_{x,x}\|=\|x\|^2$ for all $x\in H$, if and only if for every $x\neq0$ there is a $t\in S$ with $U_tx\neq0$. If E itself is locally compact and S has a non-void interior and is separable, then a similar theorem holds for the integral taken over all of E.

R. E. Fullerton (College Park, Md.)

2629:

Baum, Leonard E. Note on a paper of Civin and Yood. Proc. Amer. Math. Soc. **9** (1958), 207–208.

The author shows that some results of Civin and the

reviewer concerning commutative Banach algebras with a countable space of maximal regular ideals [Proc. Amer. Math. Soc. **7** (1956), 1005–1010; MR **18**, 586] can be improved. This is done by the utilization of an approximation theorem of Lavrent'ev in place of a more cumbersome transfinite argument. *B. Yood* (New Haven, Conn.)

2630:

Pless, Vera. The continuous transformation ring of biorthogonal bases spaces. Duke Math. J. **25** (1958), 365–371.

The dual vector spaces M and N over a division ring D are called biorthogonal bases spaces if dim M=dim N and there exist bases $\{x_i\}$ of M and $\{f_j\}$ of N such that $(x_i, f_j)=\delta_{ij}$. If dim M=dim $N=\aleph_0$, then it is known that M and N are biorthogonal bases spaces. This paper studies the ring $L(M, N)$ of all continuous (in the N-topology) linear transformations on M of such spaces. The main result is as follows: If $A\subset B$ are nonzero ideals of $L(M, N)$, then B/A is a primitive ring that is not regular and does not have minimal r-ideals.

R. E. Johnson (Northampton, Mass.)

2631:

Nelson, Edward. A functional calculus using singular Laplace integrals. Trans. Amer. Math. Soc. **88** (1958), 400–413.

For Fourier integrals $f(x)\sim\int_{-\infty}^{\infty} e^{-ix\alpha}E(\alpha)d\alpha$ suitable generalizations (in the sense of "distributions") are objects symbolically denotable by

$$f(x)\sim\int_{-\infty}^{\infty} e^{-ix\alpha}\frac{d^kE_k(\alpha)}{d\alpha^{k-1}}.$$

Now, the author analogously generalizes the Laplace integral $f(x)=\int_0^\infty e^{-tx}\mu_0(t)dt$ so that it would be denotable by

$$f(x)=\int_0^\infty e^{-xt}\frac{d^k\mu_k(t)}{dt^{k-1}}.$$

However, he does not base himself on such a symbolism or even introduce it, but is operating with the tool of functional mappings instead. He is mainly interested in the case in which x is an element of a Banach algebra, and his leading specific statement is as follows. If the algebra has a unit and $\sum_{n=1}^\infty \|(1-x)^n\|/n^{3/2}<\infty$, then x has square root x_0 representable by the absolutely convergent integral

$$x_0=-\frac{1}{2\pi^{\frac{1}{2}}}\int_0^\infty \frac{e^{-tx}-1}{t^{3/2}}dt.$$

Now if x is the infinitesimal generator of a stationary Markoff process then by "subordination" (as introduced by this reviewer) x_0 generates another process and the original relates to the new as a Gaussian process to a Cauchy process. Also the author obtains the following description in terms of path functions (in his own words). We have two independent Brownian motions ξ and ξ'. For all t, we wait until the ξ' particle first reaches t and simultaneously observe the ξ particle. Then the ξ particle appears to move according to the Cauchy law.

S. Bochner (Princeton, N.J.)

2632:

Hille, Einar. On roots and logarithms of elements of a complex Banach algebra. Math. Ann. **136** (1958), 46–57.

Let $\mathfrak{B}$ be a non-commutative Banach algebra having unit element e and let $\mathfrak{G}$ be the group of regular or invertible elements of $\mathfrak{B}$. The paper is concerned with kth roots and logarithms of elements of $\mathfrak{G}$, i.e., solutions x of $x^k=a$, and exp $x=a$. If x_1 and x_2 are distinct kth roots of a such that $\sigma(x_1)\cap\sigma(\omega^jx_2)=\emptyset$ for $j=1, \cdots, k-1$ and $\omega=e^{2\pi i/k}$ ($\sigma(b)\equiv$the spectrum of b), then x_1 and x_2 commute, and there exist k idempotents e_j, some of which may be zero, which commute with x_1 and x_2 and are such that $e_ie_j=\delta_{ij}e_j$, $\sum_j e_j=e$ and $\sum_j \omega^je_j=x_1x_2^{-1}$. The proof is based on properties of the spectrum of the operator $S(b)[x]=b^{-1}xb$ for b in $\mathfrak{G}$. Further, if $b^k=a$, and $\sigma(b)\cap\sigma(\omega^jb)=\emptyset, j=1, \cdots, k-1$, then there exists a neighborhood of a of which all points are kth powers of the elements in a neighborhood of b. The condition $\sigma(b)\cap\sigma(\omega^jb)=\emptyset$ can be weakened. Similar theorems are proved for logarithms; the operator $T(a)[x]=ax-xa$ replaces the operator $S(b)$ in their derivation.

T. H. Hildebrandt (Ann Arbor, Mich.)

2633:

Harish-Chandra. A formula for semisimple Lie groups. Amer. J. Math. **79** (1957), 733–760.

This paper is another contribution to the as yet unsolved problem of computing explicitly the Plancherel formula for semi-simple Lie groups G. It is known from the general theory of locally compact groups that such a Plancherel formula exists. Combining this with the result of the author that every unitary representation of G is of type I, it follows that the problem reduces to computing the dual measure on the space of equivalence classes of irreducible unitary representations. No general method for such a computation has as yet been discovered. For semi-simple Lie groups the problem seems to be closely related to the behavior of the conjugacy-classes of Cartan subgroups and certain functions on them.

In the present paper a formula is developed which expresses the Dirac delta measure at the unit element of G as an integral over the character groups of a maximum number $r(G)$ of non-conjugate Cartan subgroups of G. If $r(G)=1$ this formula is substantially equivalent to the Plancherel formula for G. This allows one to hope that even if $r(G)>1$ the author's formula is closely related to the Plancherel formula and may be regarded as a first step towards its derivation. Such a relation has already been recognized by Gel'fand and Graev [Dokl. Akad. Nauk SSSR. (N.S.) **92** (1953), 461–464; MR **15**, 683]. A short account of the principal results of this paper has appeared in Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 538–540 [MR **18**, 218]. *F. I. Mautner* (Paris)

2634:

Vilenkin, N. Ya. On the theory of associated spherical functions on Lie groups. Mat. Sb. N.S. **42(84)** (1957), 485–496. (Russian)

This paper contains proofs of theorems announced earlier and one additional theorem [Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 742–744; MR **19**, 153]. Our review of the author's preliminary announcement needs correction. The description which it gives of associated spherical functions refers only to a special case of the functions actually considered by him. Moreover n should be the product of the dimension of S with its multiplicity of occurence in T^H. We give now a fuller description which covers the general case.

Let H be a suitably restricted subgroup of the Lie group G and let T be an irreducible unitary representation of G in a Hilbert space $\mathfrak{H}$. Let T^1, T^2, $\cdots$ denote the distinct irreducible unitary representations of H which are components of the restriction T^H of T to H. Let $\mathfrak{H}_k$ denote the maximal closed subspace of $\mathfrak{H}$ on which T^H is a direct sum of replicas of T^k so that $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2\oplus\cdots$. Let E_k be the projection on $\mathfrak{H}_k$. Then for each pair m, k and each

$x \in G$ the restriction of $T_m E_x$ to $\mathfrak{H}_k$ is a bounded linear transformation from $\mathfrak{H}_k$ to $\mathfrak{H}_m$. These linear transformation valued functions $x \to Q_{mk}(x)$ are essentially the author's associated spherical functions. However he introduces a basis in each $\mathfrak{H}_k$ and thus obtains them as matrix valued functions. These bases are chosen so that T^H restricted to $\mathfrak{H}_k$ appears as a direct sum of identical matrix representations. *G. W. Mackey* (Cambridge, Mass.)

2635:

Tomiyama, Jun. On the projection of norm one in W^*-algebras. Proc. Japan Acad. 33 (1957), 608–612.

Theorem. Let M be a C^*-algebra with unit and N its C^*-subalgebra. If π is a projection of norm one from M to N, then 1) π is order preserving, 2) $\pi(axb)=a\pi(x)b$ for each $a, b \in N$, 3) $(\pi(x))^*\pi(x) \leq \pi(x^*x)$ for all $x \in M$. From this theorem there are systematically deduced a number of known results on conditions under which a C^*-algebra admits a faithful representation as a W^*-algebra. In addition it is shown that if M is a W^*-algebra, N a C^*-subalgebra of M, and π a projection of norm one from M to N, then 1) N is a W^*-algebra if $\pi^{-1}(0) \cap \bar{N}$ is weakly closed where $\bar{N}$ is the weak closure of N in M, and 2) N is a W^*-subalgebra if π is faithful on positive elements in M. Also an AW^* analogue is established.

S. Sherman (Philadelphia, Pa.)

2636:

Mil'man, D. P. Some theorems of non-linear functional analysis and their application in the theory of local groups. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 1(73), 222–226. (Russian)

The main object of the paper is to state (without proofs) some theorems concerning a local topological group G where G, as a space, is a subset of a Banach space E. First a natural definition of differentiability of a function $f(x)$ whose domain is an arbitrary subset G of E and whose range is in E (or in another Banach-space E_1) is given; the derivative $f'(x)$ is an operator function, i.e., for each $x \in G$, $f'(x)$ is a linear operator on E. Furthermore, corresponding to each $x \in G$ there is defined in a natural way a subset E_x of elements of E which may be considered as the "tangent space" to G at x.

Let now $G \subset E$ be a local topological group with unit element a and the composition law $x * y = F(x, y)$. It is assumed that there is at least one continuously differentiable path on G starting at a. Moreover, various assumptions concerning the derivatives of F are made; in particular $F_1'(a, x)^{-1}$ is supposed to exist where $F_1'(z, x)$ denotes the derivative of $F(z, x)$ with respect to z. If F_2' is defined correspondingly and if $\chi(x)=F_1'(a, x)^{-1}F_2'(x, a)$, then Theorems 1 and 2 contain (among others) the following statements: E_x is a linear subspace of E, and $E_x = F_1'(a, x)E_a = F_2'(x, a)E_a$; the map $x \to \chi(x)$ defines a linear representation of G with E_a as representation space; E_a becomes a Lie algebra if for u, v in E_a the commutator operation $[u, v]$ is defined as the result of the bilinear operator $\chi'(a)$ operating on uv.

Theorems 3 and 4 deal with a subset G of E which is not necessarily a group and satisfies only certain geometric conditions. Theorem 3 gives an integrability condition for an equation of the form

$$y_x'u = f(x, y)u, \quad y|_{x=a} = y_0,$$

where $f(x, y)$ is a bilinear operator depending on $x \in G$, $y \in E$ and operating on $u \in E_x$; $y(x)$ is the unknown. In the case that G is a sphere, problems of this type had been treated by Michal and Elconin [Acta Math. 68 (1937), 72–107] and M. K. Gavurin [Leningrad. Gos. Univ. Uč. Zap. 137 Ser. Mat. Nauk 19 (1950), 59–154; MR 20, 1242]. Theorem 4 deals with the condition that a certain product integral is independent of the path of integration on G.

Theorem 5 deals again with the case that $G \subset E$ is a local topological group under certain additional assumptions. If $N(x)$ ($x \in G$) is a linear representation of G with a Banach space E_1 as representation space, then a 1-1 correspondence is established between this representation and the bilinear operator $D = N'(a)$ from (E_a, E_1) to E_1. $N(x)$ is given in the form of a product integral whose integrand involves D. The operator D effects a homomorphism of the Lie algebra E_a of G into the Lie algebra of $N(G)$. *E. H. Rothe* (Ann Arbor, Mich.)

2637:

Semenov, M. P. On the question of structure of non-linear operators. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. 1 (1956), 77–79. (Russian)

The following theorem is established. Let ϕ_0 be a solution of the equation $A\phi = \lambda_0\phi$, $\lambda_0 \neq 0$. In the sphere T of radius R with center ϕ_0, let A satisfy the condition

$$\|A\phi - A\psi\|/\|\phi-\psi\| < |\lambda_0| - \gamma, \quad \gamma > 0.$$

Then the equation $A\phi = (\lambda_0 + \mu)\phi$ has a unique continuous solution in T depending on μ, with $\mu < R\gamma/(R + \|\phi_0\|)$.

An analogous theorem was formulated in an article by the reviewer [Mat. Sbornik N.S. **33(75)** (1953), 545–558; MR **15**, 719], but in the proof an important condition was overlooked, since it was implied there that the operator A gives not only topological but also open mappings. M. P. Semenov has constructed an example to show that without this hypothesis one of the reviewer's statements is false.

V. V. Nemyckiĭ (RŽMat **1957** #3228)

2638:

Krasnosel'skiĭ, M. A. On a possible generalization of the method of orthogonal trajectories. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 1(73), 160–162. (Russian)

Let $\{F(x_0, t)\}$, $0 \leq t \leq 1$, be a family of closed sets of a space R such that $F(x_0, 0) = x_0 \in R$ and $d[F(x_0, t_1); F(x_0, t_2)] \to 0$ as $t_1 - t_2$, where d is the Hausdorff distance. If for every connected set $G_1 \subset G_2$ the sets $\bigcup_{x \in G_1} F(x, t)$ are connected for every t and

$$\lim_{t_1 \to t_2} d[\bigcup_{x \in G_1} F(x, t_1);\ \bigcup_{x \in G_1} F(x, t_2)] = 0,$$

then the family $\{F(x, t)\}$ is called by the author a generalized continuous deformation of the set $G \subset R$. Let $x = x(t; x_0)$ be an integral funnel of solutions of the equation $dx/dt = \operatorname{grad} \Phi(x)$, and let $f(x_0, t)$ be the totality of all values of $x(t; x_0)$ from the integral funnel for fixed t. In the opinion of the author, it will be possible, by means of a generalization of the theorem of Kneser on the structure of an integral funnel for systems of ordinary differential equations, to prove that, with natural restrictions, $\{f(x, t)\}$ is a generalized continuous deformation on R. Two other problems are formulated.

M. M. Vaĭnberg (RŽMat **1957** #7179)

2639:

Ckiriya, K. E. On critical points of a weakly continuous functional in Banach spaces. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 24 (1957), 89–110. (Russian)

For a Fréchet differentiable function on a Banach space E, $L_f x$ denotes the gradient of $f(x)$: $Nx = \operatorname{grad} \|x\|$.

With $E=L_p$, the existence of critical points on a surface $\varphi(x)=0$ for a function f, which are the points where $L_f x=\lambda L_\varphi x$, is discussed. If $[V]$ is a homotopic equivalence class of compact sets, $c=\sup_{V\in[V]}\min\{f(x): x\in V\}$, it is shown by the method of orthogonal trajectories, i.e., by deformations along paths

$$\frac{dx}{dt}=N^{-1}L_f x-(N^{-1}L_f x, L_\varphi x)\|Lx\|^{-q}N^{-1}L_\varphi x,$$

that there is a point at which $f(x)=c$ and $L_f x=\lambda L_\varphi x$. Arguments used by Citlanadze [e.g. Trudy Moskov Mat. Obšč. 2 (1953), 235–274; MR 14, 1094] are used to show existence of a sequence of critical values corresponding to the Lyusternik-Šnirelman categories of sets in the space. Results for critical values subject to a finite set of conditions are given. *J. L. B. Cooper* (Cardiff)

2640:

Azbelev, N. V.; and Calyuk, Z. B. On Čaplygin's problem. Ukrain. Mat. Ž. 10 (1958), no. 1, 3–12. (Russian. English summary)

In this paper the following problem is considered. Let $P(y)=0$ be an operator equation, where the operator P is defined on a semiordered set X. An element $\bar{z}\in X$ is to be constructed satisfying the condition $\bar{z}\geqq y$. The problem is equivalent to that of estimating the error of an approximate solution of the equation $P(y)=0$.

The authors give the solution of the problem in semiordered normed spaces; present the necessary and sufficient condition of positivity of an inverse operator corresponding to a linear differential equation; and illustrate the general theorems by applications to nonlinear differential equations. *S. Kulik* (Logan, Utah)

2641:

Bykov, Ya. V. On the problem of existence of eigenvectors of non-linear operators. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 265–268. (Russian)

Let A be a weakly continuous non-linear operator from H to H, where H is the real separable Hilbert space. Let $\{e_k\}$ be a complete orthonormal set and c^2 a positive constant. For each $m=1, 2, 3, \cdots$, let h_m denote any vector of the form $c_1e_1+c_2e_2+\cdots+c_me_m$ with $c_1^2+c_2^2+\cdots+c_m^2=c^2$. We shall say that A satisfies condition F if, for each fixed m, there exists a functional $B(h_m)$ such that $\partial B(h_m)/\partial c_k=(Ah_m, e_k)$. Then the following two theorems are proved. i) If A satisfies condition F, then the equation $\lambda\phi=A\phi+f$ has a solution for at least one real value of λ. ii) If A satisfies condition F with $A(0)=0$, and if $\lim_{m\to\infty}B(h_m)$ exists for every sequence $\{h_m\}$ weakly convergent to zero, then the operator $A(h)$ has at least countably many real eigenvalues.

Several earlier theorems on integral equations are noted as special cases, e.g., Lotkin's theorem in Bull. Amer. Math. Soc. 50 (1944), 833–841 [MR 6, 159]. Various other such theorems are given as examples or corollaries. *S. H. Gould* (Providence, R.I.)

2642:

McFarland, J. E. An iterative solution of the quadratic equation in Banach space. Proc. Amer. Math. Soc. 9 (1958), 824–830.

The equation considered has the form

$$Bxx+Ax=y,$$

where A is a linear operator and B a bilinear operator in Banach space. The solution is that suggested by the continued fraction solution of the ordinary quadratic and is given by the sequence

$$F_0=z,\quad F_{n+1}=(A+BF_n)^{-1}y.$$

With appropriate assumptions on the operators, it is shown first that, if the sequence is defined and has a limit, the limit is a solution. Next, conditions are established for assuring that the sequence be defined, and then for assuring that it have a limit.

A. S. Householder (Oak Ridge, Tenn.)

2643:

Granas, A. Über einen Satz von K. Borsuk. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 959–962, LXXX–LXXXI. (Russian summary)

K. Borsuk [Fund. Math. 21 (1933), 236–243] proved the following theorem: let $f(x)$ be a continuous transformation of a finite-dimensional Euclidean space X into itself having the property that there exist two positive numbers η and ε such that $\|f(x)-f(y)\|<\eta$, $x, y\in X$, implies $\|x-y\|<\varepsilon$; then $f(x)$ is a mapping onto X, i.e., $f(X)=X$. The author generalizes Borsuk's theorem to an arbitrary Banach space X with the additional condition that $f(x)$ be of the form $f(x)=x-F(x)$, where $F(x)$ is a compact transformation (not necessarily linear) of X into itself. Applying this to functional equations, he shows that the well-known Fredholm alternative can immediately be deduced from this generalized theorem.

L. Brown (Detroit, Mich.)

2644:

Altman, M. On a theorem of K. Borsuk. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 1037–1040, LXXXVI. (Russian summary)

Let $F(x)$ be a continuous compact transformation (not necessarily linear) of a Banach space X into itself. The author proves that if $f(x)=x-F(x)$ satisfies the following condition: for each $x\in X$ there exist two positive numbers η_x and ε_x such that $y, z\in S(x, \varepsilon_x)$ and $\|f(y)-f(z)\|<\eta_x$ imply $\|y-z\|<\varepsilon_x$, where $S(x, \varepsilon_x)$ denotes the sphere with center x and radius ε_x — then the image $f(X)$ is an open subset of X. Furthermore, if for every neighborhood U_x of x, ε_x can be chosen so that $S(x, \varepsilon_x)\subset U_x$, then f is an open map. If F is linear, the well-known classical Fredholm alternative is obtained. *L. Brown* (Detroit, Mich.)

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See also 2591, 2639.

2645:

Goldstine, H. H. Conditions for a minimum in abstract space. Illinois J. Math. 2 (1958), 111–123.

In this paper the author proves a more general multiplier rule for abstract minimum problems than those given in his earlier papers [Bull. Amer. Math. Soc. 44 (1938), 388–394; ibid. 46 (1940), 142–149; MR 1, 146]. Let U and V be Banach spaces, and let G be real valued on an open set U_0 in U, while F has values in V for u in U_0. Suppose that F and G are of class C'' on U_0, and that u_0 minimizes G in the subset of U_0 on which F vanishes. Then there exist a constant l_0 and a linear functional L defined at least on the range S_0 of the linear mapping $dF(u_0; du)$ (the differential of F at u_0), such that $l_0dG(u_0; du)+L(dF(u_0; du))=0$ for all du in U. Also l_0 and L are not both zero, but L may fail to be continuous in case S_0 is not closed. When $S_0=V$, l_0 can be taken equal to 1,

and this is the normal case. Then it is shown in addition that the second variation at u_0 of $G(u)+L(F(u))$ is nonnegative on the subset of U where $dF(u_0; du)=0$. It is shown also that if u_0 satisfies the multiplier rule and the usual strengthened form of this last condition, then u_0 furnishes a relative minimum. The principal advance over the author's previous papers lies in omitting the assumption that the space U admits continuous projections on closed linear subspaces.

L. M. Graves (Chicago, Ill.)

2646:

Plotnikov, V. I. On the differentiability of the solutions of variational problems in a non-parametric form. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 746–749. (Russian)

Let $z_0(x, y)$ minimize the variational integral

$$\iint F(x, y, z, p, q)dxdy,\ p=z_x,\ q=z_y,$$

among all functions which are AC Tonelli in a given bounded region G, which have finite Dirichlet integral, and which satisfy given boundary conditions. Suppose also that $z_0(x, y)$ is unique in the small, and that F satisfies conditions 1)-4) listed below. Then $z_0(x, y)$ satisfies a Lipschitz condition on each closed subregion of G, from which it follows by results of Morrey [Trans. Amer. Math. Soc. **43** (1938), 126–166] and other authors that $z_0(x, y)$ has all the smoothness properties which the integrand F possesses. The conditions on F are: 1) regularity; 2) for $\sqrt{(p^2+q^2)}\equiv R\geqq\Delta>0$ we can write

$$F\equiv f(x, y, z)F^{(1)}(p, q)+F^{(2)}(x, y, z, p, q),$$

where $f>0$, $F^{(1)}$ and $F^{(2)}$ are four times differentiable; 3) $F^{(1)}$ is positively homogeneous of degree $\alpha\geqq2$ and $F_{pp}^{(1)}F_{qq}^{(1)}-(F_{pq}^{(1)})^2\geqq L>0$ for $p^2+q^2=1$;

4) $$\frac{\partial^\beta F^{(2)}(x, y, z, p, q)}{\partial p^m\partial q^n\partial z^k\partial x^l\partial y^s}\leqq L_1R^{\alpha-\gamma-m-n},$$

where $m+n+k+l+s=\beta$; $\beta=0, 1, 2, 3, 4$; $(x, y)\in G$; $|z|\leqq z_0$; $L_1=L_1(z_0)>0$; $\gamma>0$; $R\geqq\Delta$. These sufficient conditions supplement ones given by Sigalov [same Dokl. (N.S.) **85** (1952), 273–275; MR **14**, 291].

W. H. Fleming (Providence, R.I.)

2647:

Dedecker, Paul. Calcul des variations et topologie algébrique. Mém. Soc. Roy. Sci. Liège (4) **19** (1957), no. 1, 216 pp. Also published by Université de Liège, Faculté des Sciences, 1957 (200 francs belges).

The purpose of this memoir is to develop, in detail, a global theory of calculus of variations, based on modern topological notions such as jets, sheafs, spectral sequence, etc.

The first two chapters are devoted to a development of the notion of local class of maps from E to F, sheaves, in particular sheaves of germs or jets of differential forms, differential cubic homology, systems of exterior equations (with a detailed discussion of the set of zeros of such a system), complete systems, systems in involution, etc. Ch. III introduces the spectral diagram, a generalization of the spectral sequence, using the homology quotients $E(p, q, r, s)$ introduced by Deheuvels [Ann. of Math. (2) **61** (1955), 13–72; MR **16**, 1042]; for suitable sets of indices the quotients are organized into a sequence of order two, whose homology yields again certain other quotients, generalizing $E_{r+1}=H(E_r)$ for spectral sequences. In the last chapter the spectral diagram is shown to contain and to generalize the notion of integral invariant.

Ch. IV discusses, for later use, filtration on the algebra of differential forms derived from a saturated subgroup of the group of (differentiable) chains; "saturated" means roughly that the subgroup contains, along with any cube u, any "subcube" of u.

Ch. V gives a general definition of a calculus of variations problem of dimension p. In the manifold V are given subgroups A and B of the singular chain group, satisfying certain conditions (called group of restraints (condition de liaison), resp. group of boundary conditions), and a section, Ω_p, of the sheaf of germs of p-forms modulo some subsheaf orthogonal to A. A p-chain a is extremal if for any homotopy h_t the function $t\to\int_{h_t a}\Omega_p$, has derivative 0 for $t=0$. A variational structure is called free if, briefly, every p-cube in A remains in A under any homotopy for small t-values; Ω_p can then be considered as a differential form on V. For free problems the formalism of the first variation is developed in all generality, including transversality. A lifting of a variational structure in V consists in such a structure in another manifold $\bar V$ together with a map f: $\bar V\to V$ relating the two structures in a certain way; roughly speaking, the extremals in $\bar V$ are the chains that project into the extremals of V. The "fundamental problem" is the construction of free liftings. The existence of free semi-liftings (a weakened form of lifting) is proved for a large class of problems; the $\bar V$ are suitable fiber spaces over the V; the construction amounts to a global form of Lagrange multipliers. The classical variational problems of dimension p and order k appear as problems in the associated manifold of $(p-k)$-contact elements. Free liftings are constructed for them (Ch. VI). The congruences of Lepage find an interpretation here, as does the concept of normality (Bliss, etc.); many other ramifications of the theory are discussed. Part of the material was announced earlier [e.g., Géométrie différentielle, Colloq. Internat. Centre Nat. Rech. Sci., Strasbourg, 1953, pp. 17–34; Convegno Internazionale di Geometria Differenziale Italia, 1953, Edizioni Cremonese, Roma, 1954, pp. 247–262; IIIe Congrès Nat. Sci., Bruxelles, 1950, v. 2, Fédération Belge Soc. Sci., Bruxelles, pp. 29–35; MR **16**, 50, 521; **17**, 45].

H. Samelson (Ann Arbor, Mich.)

2648:

Pitcher, Everett. Inequalities of critical point theory. Bull. Amer. Math. Soc. **64** (1958), 1–30.

The first part of the paper gives a notably compact and lucid account of the critical point theory of a real-valued function on a Riemannian manifold, keeping the notion of homotopy equivalence in the foreground and pointing out the role of the results. Examples are given indicating lack of sharpness of the Morse inequalities in various situations. These inequalities relate the ranks of certain relative Betti groups. The latter part of the paper presents the main novelties, namely a sharpened form of the Morse inequalities taking torsion into account, and may be characterized as the replacement of fields by, for instance, the integers as coefficients. A direct approach through the universal coefficient theorem is presented with the coefficient field that of the residue classes mod p, p a judiciously chosen prime dividing the relevant torsion coefficients. This is satisfactory when no critical point is degenerate. For the more general situation the author introduces a chain subcomplex Q of a chain complex C called a skeleton of C, such that the inclusion map of Q into C induces an isomorphism onto of the homology groups. For singular homology C is a free chain complex and so is Q, and in the key application $Q=B_k+F_k+F_{k-1}$, where B_k is a free group of rank R_k (the Betti number in dimension k of C), F_k has rank n_k, the number of torsion

coefficients, and the boundary operator is suitably defined. The sharpened inequalities are derived by taking ranks of the incidence matrices of Q_{k+1}, Q_k. Analogously to the classical index situation for fixed points on a complex, $R_k+n_k+n_{k-1}$ is shown to be the minimum number of critical points for functions with non-degenerate critical points only, which are close to f in the sense of a natural metric. *D. G. Bourgin* (Urbana, Ill.)

GEOMETRIES, EUCLIDEAN AND OTHER

See also 2669, 2671, 2772.

2649:

Zimmer, Hans-Georg. Bemerkungen zum Trapezdesargues. Math. Ann. **135** (1958), 274–278.

In an affine plane the "theorem of the Desargues trapezoids", as the author calls it (where for two triangles ABC, $A'B'C'$, from AA', BB', CC' intersecting in a point P and $AB\|A'B'$, $AC\|A'C'\|BB'$ follows $BC\|B'C'$), is shown in its correlation with the "parallelogram-Desargues" (the special case of the above theorem where $BC\|AA'$), the theorem of the midpoint and the little Desargues theorem. The reasoning culminates in the proof that the theorem of the Desargues trapezoids, valid along two parallel lines g and g', makes the little Desargues theorem valid for that direction. Its validity in one direction and along one more line induces its validity along any line of the affine plane. The work is based on publications by R. Moufang [e.g. Math. Ann. **105** (1931), 536–601; **106** (1932), 755–795; **110** (1934), 416–430].
S. R. Struik (Belmont, Mass.)

2650:

Lingenberg, Rolf. Euklidische Pseudoebene über einer metrischen Ebene. Abh. Math. Sem. Univ. Hamburg **22** (1958), 114–130.

A set of "points" and "lines", together with two binary relations "incidence" (between points and lines) and "perpendicularity" (between lines), is called a "metric plane" provided the following axioms are satisfied. I (Incidence): (Ia) There exists at least one line and a point not on this line. On each line there are at least three points. (Ib) There is exactly one line incident with each two points. II (Perpendicularity): (IIa) Symmetry. (IIb) Perpendicular lines meet. (IIc) Through a point there is at least one line perpendicular to a given line, and if the point is on the line exactly one. Definition: A 1-1 involutorial mapping of the points and lines which preserves incidence and perpendicularity and which leaves all the points of one line (and only these) fixed is a "reflection in the line". III (Reflection): (IIIa) To each line g corresponds at least one reflection in g. (IIIb) If a, b, c are lines through the point P (or if they are perpendicular to a line g), then there exists a line d through P (or perpendicular to g) such that a sequence of reflections about a, b, and c is equivalent to one about d.

A metric plane is called a "euclidean plane" provided: (IVa) There exists four lines a, b, c, d with a and b perpendicular to c and d; (IVb) two lines either meet or have a common perpendicular. Reflections in a line g are shown to be unique and denoted by σ_g.

In this note it is shown that any metric plane can be extended (in a very special way) to a euclidean plane called here an associated euclidean pseudoplane. To this end a ternary equivalence relation is introduced, three lines a, b, c being in the same class (bundle) if $\sigma_a\sigma_b\sigma_c=\sigma_x$ for some line x. The resulting bundles are then called "ideal points". "Ideal lines" are then defined as collections of ideal points, a "proper ideal line" being first defined as a collection of bundles having a common line. This definition is then extended by means of a transformation of the original metric plane called a "partial rotation" about a fixed point O, and finally the set of bundles defined by the lines perpendicular to those through O is also called an ideal line o. The points and lines thus defined together with the obvious incidence relation are then shown to be a projective plane. The plane is made affine by singling out the line o, and finally appropriate definitions of perpendicularity and reflection convert this into a euclidean plane, the induced relations being intimately related of course to their correspondents in the original metric plane.

The principal theorem reads as follows: Each metric plane may be imbedded in a euclidean pseudoplane A such that the perpendicularity in the metric plane coincides with the induced orthogonality in the euclidean pseudoplane in a point O and such that the affinities in A induced by the reflections in the metric plane about lines through O agree with the euclidean pseudo-reflections about lines in $\bar{O}$ (the correspondent of O in A).

{The reading of the paper would be greatly facilitated by reference to the forthcoming book of F. Bachmann [Aufbau der Geometrie aus dem Spiegelungsbegriff, Grundlehren Math. Wiss. in Einzeldarst., Springer, Berlin], to which the author frequently alludes.}
L. M. Kelly (East Lansing, Mich.)

2651:

Pišl, M. Curves in the Gauss plane. I, II, III. Pokroky Mat. Fys. Astr. **2** (1957), 4–13; 144–156; 271–284. (Czech)

The author presents an account of plane coordinate geometry in which the Cartesian coordinates x, y are eliminated in favour of $z=x+iy$, $\bar{z}=x-iy$. Part I deals with the straight line, Part II with the conic sections in standard forms, and Part III with the general equation of the second degree. Homogeneous coordinates are also considered. *F. V. Atkinson* (Canberra)

2652:

Hirano, Kotarō. On some center circles and their relations. Sûgaku **8** (1956/57), 210–211. (Japanese)

Theorem 1. Let a_i ($i=1, 2, \cdots, n$) be n straight lines on a plane; $A(ij)$ the intersection of a_i, a_j; $A(ijk)$ the circumcircles of the tri-sides $a_ia_ja_k$; and let generally $A(12\cdots n)$ be the center circle with respect to the n straight lines in Steiner's sense. (i) The case $n=4$. If we draw the perpendicular from the intersection of the parallel through $A(ij)$ ($i, j=1, 2, 3$; $i\neq j$) to a_4 with the circumference of the circle $A(123)$, on a_4, then three such perpendiculars are concurrent in $A(123)$, which we denote by Q_{4-123}. We have four such points Q_{1-234}, Q_{2-341}, Q_{3-412}, Q_{4-123}; and they lie on a circle, which we denote by $B(1234)$. (ii) The case $n=5$. If we draw the perpendicular from the intersection of the parallel drawn from the center of $A(ijk)$ ($i, j, k=1, 2, 3, 4$; $i\neq j\neq k\neq i$) to a_5, with the circumference of the circle $A(1234)$, on a_5; then four such perpendiculars are concurrent in $A(1234)$, which we denote by Q_{5-1234}. We have five such points Q_{1-2345}, Q_{2-3451}, Q_{3-4512}, Q_{4-5123}, Q_{5-1234} and they lie on a circle, which we denote by $B(12345)$. (iii) And so on. The circles $A(12\cdots n)$ and $B(12\cdots n)$ are concentric and they coincide when $n=4$.

Theorem 2: Consider another straight line a_0. (i) The

case $n=4$. In accordance with Theorem 1, we have four points Q_{0-234}, Q_{0-341}, Q_{0-412}, Q_{0-123} and they lie on a circle, which we denote by $C(1234)$. (ii) The case $n=5$. We have five points Q_{0-2345}, $\cdots$, Q_{0-1224} and they lie on a circle, which we denote by $C(12345)$. (iii) And so on. The point $Q_{0-12\cdots n}$ is the center of the circle $C(12\cdots n)$.

Theorem 3: The $(n+2)$ circles $A(23\cdots n), \cdots, A(12\cdots n-1)$, $B(12\cdots n)$ and $C(12\cdots n)$ pass through one and the same point and also the n circles $C(23\cdots n)$, $\cdots$, $C(12\cdots n-1)$ pass through one and the same point.
T. Takasu (Yokohama)

2653:

Hirano, Kotaro. On certain kinds of point sequences. II. Sûgaku 9 (1957/58), 150–151. (Japanese)

[For Part I, see Sûgaku 6, no. 4 (1955).]

Lemma: Let the inscribed circle with center O of the triangle ABC touch BC, CA, AB in D, E, F respectively. Take the points P, Q, R respectively on OD, OE, OF so that $OP=OQ=OR$, which are positive or negative according as they have the same or opposite sense with OD, OE, OF respectively. Let the feet of the perpendiculars drawn from A, B, C upon an arbitrary straight line p passing through O be A', B', C' respectively. Then PA', QB', RC' are concurrent. (This lemma is proved by means of complex numbers representing D, E, F on the circle O.)

Theorem (stated in Part I, p. 219): Let three straight lines a_1, a_2, a_3 touch a given circle O in A_1, A_2, A_3 respectively. The three straight lines joining the feet of the perpendiculars drawn from (a_2, a_3), (a_3, a_1), (a_1, a_2) upon a straight line p passing through O to the three points, which are symmetric to A_1, A_2, A_3 respectively with respect to the center O, meet in a point x_{123} on the circumference of the circle O. Let a_4 be another straight line touching the same circle O in A_4. Let the point related to a_i, a_j, a_k analogously to x_{123} for a_1, a_2, a_3, be x_{ijk}. The straight lines A_1x_{234}, A_2x_{341}, A_3x_{412}, A_4x_{123} meet in a point x_{1234} on p. Let a_5 be another straight line touching the same circle O in A_5. Let the point related to a_i, a_j, a_k, a_l analogously to x_{1234} for a_1, a_2, a_3, a_4 be x_{ijkl}. The five straight lines B_1x_{2345}, B_2x_{3451}, B_3x_{4512}, B_4x_{5123}, B_5x_{1234}, where B_1, B_2, B_3, B_4, B_5 are the points situated symmetrically to A_1, A_2, A_3, A_4, A_5, respectively, with respect to the point O, meet in a point x_{12345} on the circumference of the same circle. And so on. (This theorem is proved utilizing the above lemma.)

If we take the unit circle with center at the origin for the circle C, we have $x_{12\cdots n}=(-1)^{t}\cot(\sum\arctan)-1({}^{t}t_k)$, where $x_{12\cdots n}$ are the coordinates of $X_{12\cdots n}$ and t_k the coordinates of A_k.
T. Takasu (Yokohama)

2654:

Hirano, Kotaro. On an analogue to Kantor's theorem and a center circle. Sûgaku 9 (1957/58), 150. (Japanese)

Theorem: Let a_i ($i=1, 2, \cdots, 7$) be seven straight lines tangent to a Steiner's hypocycloid and let a_0 be any other straight line. The six points $Q_{0-23456}$, $\cdots$, $Q_{0-12345}$ [see two papers reviewed above] lie on one and the same straight line, which we denote by $L(123456)$. The seven straight lines $L(234567)$, $\cdots$, $L(123456)$ are tangent to another Steiner's hypocycloid.
T. Takasu (Yokohama)

2655:

Teuffel, Erich. Eine Eigenschaft der Quadratwurzelschnecke. Math.-Phys. Semesterber. 6 (1958), 148–152.

Let B_0, B_1, B_2, $\cdots$ be a sequence of coplanar points with $B_iB_{i+1}=1$ ($i\geqq 0$); $\angle B_iB_0B_{i+1}=\pi/2$ ($i\geqq 1$); and B_{i+1} and B_{i-1} on opposite sides of B_0B_i, $i>1$. The author uses trigonometric, algebraic and number theoretic lemmas to establish that no three distinct points B_0, B_n, B_m or B_m, B_{m+1}, B_n are collinear.
L. Moser (Edmonton, Alta.)

2656:

Toscano, Letterio. Su dei punti legati ad una involuzione di Möbius. Matematiche, Catania 11 (1956), 175–182 (1957).

Consider the two inversions determined by the two orthogonal circles: the circumcircle (O, R) of a given triangle (T), and the imaginary circle $(K, \rho_2\sqrt{-1})$ whose center is the Lemoine point K of (T) and whose radius is $\rho_2\sqrt{-1}$, where ρ_2 is the radius of the second Lemoine circle of (T). The product of these two inversions constitutes a Moebius involution. These three transformations have been used by the author in an earlier work [Bull. Soc. Roy. Sci. Liège 21 (1952), 557–573; 22 (1953), 47–65; MR 14, 895] to study some well-known points associated with a triangle, like the Brocard points, the vertices of the second Lemoine triangle, etc., and also some new points introduced by the author.

The present paper supplements the above works, and in particular calls attention to the pole of the Lemoine axis with respect to the Brocard circle. The paper concludes with a number of formulas involving the two segments t, t' which Cavallaro called the Torricelli segments [Mathesis 52 (1938), 174–175].
N. A. Court (Norman, Okla.)

2657:

Toscano, L. Sur les cercles de Schoute. Bull. Soc. Roy. Sci. Liège 26 (1957), 314–324.

Given a triangle (T), denote by O, R the center and the radius of the circumcircle, K the Lemoine point, W, W' the isodynamic points, t, t' the Torricelli segments [see the preceding review].

The Apollonian circle (Z_ν) which has for ends of a diameter a pair of points X_ν, Y_ν dividing the segment WW' harmonically in the ratio $(t'/t)^\nu$, where ν is an arbitrary number, generates a pencil of circles having W, W' for its limiting points. This pencil is identical with the one obtained by P. H. Schoute, using a different approach [Versl. Med. Akad. Wetensch. Amsterdam. Afd. Natuurk. (3) 3 (1887), 39–62].

The author derives formulas, in terms of t, t', ν, R, for the radius Z_ν of (Z_ν), the segments OX_ν, OY_ν, KX_ν, KY_ν, for integral values of ν both positive and negative; he establishes relations between circles corresponding to particular values of ν, and obtains many other results of this kind.
N. A. Court (Norman, Okla.)

2658:

Harant, M. Kotiert-axonometrische Abbildungsmethode im vierdimensionalen Raum. Publ. Fac. Sci. Univ. Masaryk 1956, 455–485. (Czech. Russian and German summaries)

There are various methods of dealing with the descriptive geometry of the four-dimensional metric space. For the understanding of such a method it is sufficient to enumerate its basic principles which govern the projection of a generic point P. These principles are: 1) A generic point P is projected perpendicularly to our three-space R_3 into P_R in R_3; 2) its distance d_P from R_3 is given; 3) R_3 is projected axonometrically on a plane π, and P_π is the usual axonometric image of P_R. The point P is uniquely given by P_R, P_π, d_P.
V. Hlavatý (Bloomington, Ind.)

2659:

Harant, M. Eine klinogonale Abbildungsmethode im Raume E_4. Acta Fac. Nat. Univ. Comenian. Math. 2 (1958), 193–217. (Slovak. Russian and German summaries)

This is a continuation of a previous paper [see above]. Here the author uses the klinogonal axonometry. This method is closely connected with the reviewer's methods, in particular with the second, where the "centers of projection" are two skew ideal lines [Časopis Pest. Mat. Fys. **52** (1923), 1–23; Enseignement Math. **24** (1924/25), 276–286]. *V. Hlavatý* (Bloomington, Ind.)

2660:

✱Stark, Marceli. Geometria analityczna. [Analytic geometry.] 2nd ed., revised. Biblioteka Matematyczna, Vol. 17. Państwowe Wydawnictwo Naukowe, Warsaw, 1958. 416 pp.

No previous knowledge of the subject is assumed on the part of the users of this book which is supposed to serve as a text in a first year university course for students of average ability. The gifted student is supposed to forage for himself and he is offered the hint that he may be interested in K. Borsuk's beautiful work "Analytic geometry in n dimensions" [Monograf. Mat. vol. 12, Warszawa-Wrocław, 1950; MR **12**, 630].

The first chapter is devoted to elementary vector theory and then vectors are used throughout the book. Two- and three-dimensional geometry are treated concurrently.

The first two thirds of the book deal with metrical and affine geometry, and this is the part considered essential for first year students. The last part of the book is given largely to projective geometry. Topics deemed of lesser importance, like pencils of circles, stereographic projections, or the details of the metrical classification of quadric surfaces, are printed in small type.

Worked out examples illustrating the theory are to be found frequently. There are no examples or exercises to be worked by the students.

The book concludes with a comprehensive index and a detailed table of contents. *N. A. Court* (Norman, Okla.)

2661:

Subramanyam, S. S. Acute angle in analytical geometry. Math. Student **25** (1957), 32–39.

Betweenness for three points on a line in euclidean geometry is replaced by separation of two point pairs in projective geometry where the straight line is a closed curve. Theorems stated about two separating (non-separating) point pairs, using cross ratios, are applied to the totality of lines through a point in a projective domain.

The acute (obtuse) angle between lines OA, OB is defined as the totality of lines such that any of them and one of the perpendiculars to OA or OB separates (or does not separate) OA, OB. The conditions that a given line $y=mx$ may lie in the acute or obtuse angle between $y=m_1x$ and $y=m_2x$ are tabulated and the same is done for the bisectors of their angles.

S. R. Struik (Cambridge, Mass.)

2662:

Schwerdtfeger, Hans. Zur Geometrie der Möbius-Transformation. Math. Nachr. **18** (1958), 168–172.

Projectivities between straight lines in the conformal plane and their representation as Möbius transformations are considered. Every perspectivity can be represented by a Möbius transformation. A Möbius transformation f represents a perspectivity if and only if the two fixed points of f and the two poles of f form the pairs of opposite vertices of a proper parallelogram. Every Möbius transformation represents a projectivity between two straight lines. Different types of Möbius transformations and their corresponding projectivities are studied in the paper.

R. Artzy (Haifa)

2663:

Skornyakov, L. A. Homomorphisms of projective planes and T-homomorphisms of ternaries. Mat. Sb. N.S. **43(85)** (1957), 285–294. (Russian)

A T-homomorphism of a ternary ring M_1 onto a ternary ring M_2 is a mapping $a\to a^\theta$ of the elements of M_1 onto those of M_2, including ∞, such that the following 7 properties hold: 1) If a^θ, m^θ, $b^\theta\neq\infty$, then $(a\cdot m\circ b)^\theta= a^\theta\cdot m^\theta\circ b^\theta$; 2) if $a^\theta=\infty$, $b^\theta\neq\infty$, $(a\cdot m\circ b)^\theta\neq\infty$, then $m^\theta=0$; 3) if $m^\theta=\infty$, $b^\theta\neq\infty$, $(a\cdot m\circ b)^\theta\neq\infty$, then $a^\theta=0$; 4) if $b^\theta=\infty$, $(a\cdot m\circ b)^\theta\neq\infty$, then either $a^\theta=\infty$, or $m^\theta=\infty$; 5) if $c=a\cdot m\circ b=a\cdot n\circ 0$, $a^\theta=c^\theta=\infty$, $b^\theta\neq\infty$, then $m^\theta=n^\theta$; 6) if $m^\theta=b^\theta=\infty$, $c=a\cdot m\circ b$, $c^\theta\neq\infty$, $0=d\cdot m\circ b$, then $a^\theta=d^\theta$; 7) if $a^\theta=m^\theta=b^\theta=c^\theta=\infty$, where $c=a\cdot n\circ 0= a\cdot m\circ b$, $0=d\cdot m\circ b$, then either $n^\theta=\infty$ or $d^\theta=\infty$.

The paper relates T-homomorphisms of ternary rings to homomorphisms of projective planes and shows that if M_1 is the ternary ring of a plane π_1 in terms of a quadrilateral $XYOI$, and if φ is a homomorphism of π_1 onto π_2 which maps $XYOI$ onto a quadrilateral $X'Y'O'I'$ which determines the ternary ring M_2 for π_2, then φ leads to a T-homomorphism of M_1 onto M_2.

Marshall Hall, Jr. (Columbus, Ohio)

CONVEX SETS AND DISTANCE GEOMETRIES

See also 2408, 2721, 2728, 2772, 2773.

2664:

Harrop, R.; and Rado, R. Common transversals of plane sets. J. London Math. Soc. **33** (1958), 85–95.

Let A_i $(i=1, 2, \cdots, n)$ denote plane point sets and A_i' the convex hull of A_i. Let T_{rn} $(r\leqq n)$ denote the set of all systems $(A_0, A_1, \cdots, A_{n-1})$ such that corresponding to every choice of indices α_ρ for $\rho<r$, satisfying (1) $\alpha_0<\alpha_1<\cdots<\alpha_{r-1}<n$, there exists a transversal g such that $g\cdot A_{\alpha_\rho}'\neq 0$ for all $\rho<r$; and let S_{rn} denote the set of all systems $(A_0, A_1, \cdots, A_{n-1})$ such that corresponding to every set of indices α_ρ satisfying (1), there is a permutation $\rho\to\pi(\rho)$ of $(0, 1, 2, \cdots, r-1)$ such that

$$(B_0+B_1+\cdots+B_{\mu-1})'\cdot(B_\mu+\cdots+B_{r-1})'=0,$$

where $B_\rho=A_{\alpha\pi(\rho)}$ for $\rho<r$. As a complement to a result of B. Grünbaum [Pacific J. Math. **5** (1955), 351–359; MR **17**, 185] the authors prove the following theorem: α) If $n\geqq 4$, then $S_{nn}T_{3n}\subset S_{n-1,n}T_{3n}\subset\cdots\subset S_{4n}T_{3n}\subset S_{4n}T_{4n}\subset S_{3n}T_{4n}\subset T_{nn}$; β) $S_{nn}T_{2n}\not\subset T_{3n}$ $(n\geqq 3)$; $S_{3n}T_{3n}\not\subset T_{4n}(n\geqq 4)$; $S_{2n}T_{n-1,n}\not\subset T_{nn}(n\geqq 3)$.

As the authors have noted [J. London Math. Soc. **33** (1958), 252] these results are closely related to those obtained by N. H. Kuiper [Nederl. Akad. Wetensch.=Indag. Math. **19** (1957), 272–283; MR **19**, 762].

L. A. Santaló (Buenos Aires)

2665:

Böhme, Walter. Ein Satz über ebene konvexe Figuren. Math.-Phys. Semesterber. **6** (1958), 153–156.

It is shown that in every closed convex planar curve

one can inscribe infinitely many pentagons which are affine equivalent to a regular pentagon.

L. Moser (Edmonton, Alta.)

2666:

Sancho de San Román, J. A new concept of affine breadth of oval bodies. Rev. Acad. Ci. Madrid **51** (1957), 229–244. (Spanish)

The affine width of a n-dimensional oval K at the point P is defined as the volume of the n-dimensional simplex with maximal volume inscribed in K with a vertex at P. Some properties of the n-dimensional ellipsoids E related with their (constant) width are given. For instance, if k_i $(i=1, 2, \cdots, n+1)$ are the gaussian curvatures of E at the vertices of an inscribed simplex of maximal volume, then the relation $\sum k_i^{-2/(n+1)}=$constant, holds. It seems that it is not known if the condition of constant affine width characterizes the ellipsoids if $n>2$. For $n=2$ analogous questions were previously considered by the author [Rev. Mat. Hisp. Amer. (4) **16** (1956), 151–171; MR **18**, 505.]

L. A. Santaló (Buenos Aires)

2667:

Santaló, L. A. On the mean curvatures of a flattened convex body. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **21** (1956), 189–194 (1957). (Turkish summary)

Let K^r be a convex body contained in an r-dimensional linear subspace L^r of Euclidean n-space E^n. Let, further, S^n and S^r be the unit spheres of E^n and L^r, respectively. Consider the mixed volumes $W_p{}^n$ of S^n taken $n-p$ times and K^r, as a subset of E^n, taken p times, $0\leqq p\leqq n$; on the other hand, consider the analogous mixed volumes $W_p{}^r$ of S^r and K^r as a subset of L^r, $0\leqq p\leqq r$, and define $W_p{}^r=0$ for $p<0$. The author proves relations which may be written $\binom{n}{p}W_p{}^n/\kappa_p=\binom{r}{p-n+r}W^r_{p-n+r}/\kappa_{p-n+r}$, where κ_p denotes the volume of the unit sphere in p-space, $\kappa_0=1$. Making assumptions of smoothness, the author actually proves these relations in terms of the mean curvatures $M_q{}^r=rW^r_{q+1}$, that is, apart from a factor depending only on r and q, the surface integrals of the elementary symmetric functions of the principal curvatures of the boundary of K^r. The $M_q{}^n$ are defined as limits of the $M_q{}^n$ for a parallel body of K^r in E^n. An integral geometric interpretation of the formulae is given.

W. Fenchel (Copenhagen)

2668:

Dinghas, Alexander. Über eine Klasse superadditiver Mengenfunktionale von Brunn-Minkowski-Lusternikschem Typus. Math. Z. **68** (1957), 111–125.

Let $\mathscr{B}$ and $\mathscr{P}(M)$ be the families of all nonvoid bounded subsets of the n-dimensional Euclidean space and of all positive bounded functions defined on $M\in\mathscr{B}$, respectively. Then the functional $I(f, M)$ on $\mathscr{P}(M)\times\mathscr{B}$ is defined as the least upper bound of Lebesgue integrals of all simple nonnegative functions on M which are $\leqq f$ and which assume their positive values on closed sets. For A_1, $A_2\in\mathscr{B}$ let A_0 be the set of all vector sums x_1+x_2 with $x_i\in A_i$. The author proves: For all $A_i\in\mathscr{B}$, $f_i\in\mathscr{P}(A_i)$, every positive number r and every $f_0\in\mathscr{P}(A_0)$ such that $f_0(x)\geqq\sup\{[f_1(x_1)^{1/r}+f_2(x_2)^{1/r}]^r: x_i\in A_i, x_1+x_2=x\}(i=1,2)$, we have

(1) $I(f_0, A_0)^{1/(r+n)}\geqq I(f_1, A_1)^{1/(r+n)}+I(f_2, A_2)^{1/(r+n)}$.

Clearly, $I(1, M)$ is the inner Lebesgue measure of M. Therefore, if $r\to 0$, $f_1=f_2=1$, (1) yields the inequality of Brunn-Minkowski-Lusternik. The relationship between (1) and results by R. Henstock and A. M. Macbeath [Proc. London Math. Soc. (3) **3** (1953), 182–194; MR **15**, 109] is discussed and an application of (1) is given.

H. M. Schaerf (Madison, Wis.)

2669:

Fejes Tóth, L.; und Molnár, J. Unterdeckung und Überdeckung der Ebene durch Kreise. Math. Nachr. **18** (1958), 235–243.

The authors obtain some new results along the lines of Fejes Tóth's "Lagerungen in der Ebene, auf der Kugel und im Raum" [Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953; MR **15**, 248; pp. 58–79]. They consider packings of non-overlapping circles in the Euclidean plane, and coverings of the plane by overlapping circles. In terms of h, the ratio of the areas of the smallest and largest circles, they define $d(h)$ as the greatest possible density for a packing and $D(h)$ as the smallest possible density for a covering, so that $d(0)=D(0)=1$, $d(1)=0.9069\cdots$ and $D(1)=1.2092\cdots$. Examples with $0<h<1$ are provided by the incircles and circumcicles of the cells of the uniform tessellation of squares and octagons:

$$0.92015\cdots\leqq d(3-2\sqrt{2})\leqq 0.92084\cdots,$$
$$1.1875\cdots\leqq D(1-\sqrt{\tfrac{1}{2}})\leqq 1.18976\cdots.$$

H. S. M. Coxeter (Toronto, Ont.)

2670:

Norlander, Göte. A covering problem. Nordisk Mat. Tidskr. **6** (1958), 29–31, 56. (Swedish. English summary)

Let $I_1, \cdots, I_n$ be closed intervals on the real axis. There is a subset $I_{n_1}, \cdots, I_{n_m}$ of disjoint I_i such that

$$\sum |I_{n_i}|\geqq 2^{-1}|\mathbf{U} I_j|,$$

where $|\,|$ means measure. If the I_j are homothetic squares in E^2, then there are disjoint I_{ni} such that

$$\sum |I_{n_i}|\geqq 4^{-1}|\mathbf{U}\, I_j|.$$

The factor 2^{-1} is sharp in the first case, whereas 4^{-1} is conjectured as the best factor in the second case, but this is proved only for congruent I_j. The result on congruent I_j was previously obtained by R. Rado [Proc. London Math. Soc. (2) **51** (1950), 232–264; MR **11**, 51].

H. Busemann (Cambridge, Mass.)

2671:

Tammi, Olli. On parallel projection of a rectangular space coordinate system on to a plane. Ann. Acad. Sci. Fenn. Ser. A. I, no. **254** (1958), 24 pp.

Let a_ν $(\nu=1, 2, 3)$ be three coplanar vectors of lengths e_ν making angles ϕ_ν $(\sum\phi_\nu=2\pi)$. According to Pohlke's Theorem, they can be derived from three mutually orthogonal vectors of equal length e, say, by parallel projection in a direction making an angle θ, say, with the normal to the plane. The author proves that

$$s_2^2s_3^2\sin^2\phi_1+s_3^2s_1^2\sin^2\phi_2+s_1^2s_2^2\sin^2\phi_3= s_1^2+s_2^2+s_3^2-1=\sec^2\theta,$$

where $s_\nu=e_\nu/e$.

Setting $n=3$ and $s=2$ in some results of Hadwiger [Comment. Math. Helv. **13** (1940), 90–107; MR **2**, 260; p. 98], one finds that, when the projection is orthogonal (so that $\theta=0$), $\sum a_\nu{}^2=2e^2$, $\sum_\nu (a_\nu\cdot a_k)^2=e^2a_k{}^2$. Thus, $e_1{}^2+e_2{}^2+e_3{}^2=2e^2$ and (setting $k=1$) $e_1{}^2+e_2{}^2\cos^2\phi_3+e_3{}^2\cos^2\phi_2=e^2$. Eliminating e^2, one obtains $e_1{}^2+e_2{}^2\cos 2\phi_3+e_3{}^2\cos 2\phi_2=0$.

The author points out that, when the ϕ's are all obtuse, there exists a triangle having sides $e_\nu{}^2$ and angles $2\phi_\nu-\pi$. Moreover, the internal bisectors of these angles give the directions of the three vectors a_ν. {The reviewer

has taken the liberty of writing ϕ_1, ϕ_2, ϕ_3 in place of the author's ϕ_2, ϕ_3, ϕ_1.} *H. S. M. Coxeter* (Toronto, Ont.)

2672:

Nitka, W. Bemerkungen über nichtisometrische Abbildungen. Colloq. Math. 5 (1957), 28–31.

Sharpening a result of Freudenthal and Hurewicz [Fund. Math. 26 (1936), 120–122], the author proves a theorem about the compensating of contractions by dilatations: Let X be a totally bounded space and for any ε let n_ε be the minimal cardinality of an ε-net of X with more than two points. Let $f(X)\subset X\subset\overline{f(X)}$. For some a, b let $\rho(a, b)=\rho(fa, fb)+12\varepsilon$. Then there are a', b', fa', fb', such that $\rho(a', b')<\rho(fa', fb')-\varepsilon(n_\varepsilon/n_\varepsilon-1)$. ($f$ may be multivalent.) — Two problems are raised.
H. Freudenthal (Utrecht)

GENERAL TOPOLOGY, POINT SET THEORY

See also 2354, 2419, 2559, 2610, 2611, 2699, 2980.

2673:

Gladkiĭ, A. V. Rarefied classes of sets admitting F_σ coverings. Mat. Sb. N.S. 44(86) (1958), 287–295. (Russian)

This paper continues the study of rarefied classes of sets, begun by A. A. Lyapunov [Mat. Sb. N.S. 24(66) (1949), 119–127: MR 10, 518]. Notation and terminology are as in the review cited. Let I be the real line. A rarefied class Ξ is said to admit F_σ coverings if every set in Ξ is contained in an F_σ that belongs to Ξ. For a closed set F, let Ξ_F be the class of all subsets of F that are of the first category in F. Theorem: A rarefied class Ξ admitting F_σ coverings can be represented in the form $\bigcap_{\alpha<\gamma}\Xi_{F_\alpha}$, where $\{F_\alpha\}_{\alpha<\gamma}$ is a transfinite well-ordered descending family of closed sets such that $F_{\alpha+1}$ is nowhere dense in F_α. The converse also holds. *E. Hewitt* (Seattle, Wash.)

2674:

Kapuano, Isaac. Classification des points d'un continu cartésien. C. R. Acad. Sci. Paris 245 (1957), 1866–1868.

Thirteen statements about continua, of which the first three seem to be true. R. H. Bing, in a letter to the reviewer, points out that statement 4 contradicts an example by Whyburn and another by Anderson, and he supplies a simple counterexample to statement 5. The remaining eight statements seem at best conjectural.
L. Gillman (Princeton, N.J.)

2675:

Anderson, R. D. A characterization of the universal curve and a proof of its homogeneity. Ann. of Math. (2) 67 (1958), 313–324.

The Sierpiński universal curve is characterized by showing that it is a C-set and that any two C-sets are homeomorphic. By methods used to establish these theorems, it is shown that the universal curve, and hence any C-set, is n-point homogeneous. That is, if $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are $2n$ points of the universal curve, then there exists a homeomorphism π of the universal curve onto itself such that $\pi(a_i)=b_i$ for $i=1, 2, \cdots, n$. The author announces that he can prove that the universal curve admits monotone open maps onto all locally connected continua such that all inverses of points are themselves universal curves.

The universal curve is the 3-dimensional analogue in the unit cube of the Cantor set in the unit interval. Every 1-dimensional locally connected continuum can be imbedded in it. According to the principal theorem of the paper, a C-set is any set homeomorphic to the universal curve. The working definition of a C-set is too long to include in a review, but it has the advantage of providing a useful tool for proving the homogeneity result and other properties of the universal curve [cf. the following review]. *W. R. Utz* (Columbia, Mo.)

2676:

Anderson, R. D. One-dimensional continuous curves and a homogeneity theorem. Ann. of Math. (2) 68 (1958), 1–16.

Using the results of an earlier paper [cf. the preceding review], the author secures characterizations of the universal curve. Let $\mathfrak{K}$ be the class of all 1-dimensional compact metric locally connected continua up to a topological equivalence. Let $\mathfrak{M}$ be the subclass of $\mathfrak{K}$ consisting of those elements of $\mathfrak{K}$ having no local cut points. The universal curve is an element of $\mathfrak{M}$. In addition to the universal curve, the two types of primitive skew curves [C. Kuratowski, Fund. Math. 15 (1930), 271–283] are specific topological curves needed to describe the results. The primitive skew curve of Type 1 consists of two disjoint sets of three vertices each and nine arcs joining the vertices from different sets in pairs so that the arcs are disjoint except for their endpoints. This is the curve required for the gas-water-electricity problem. The primitive skew curve of Type 2 consists of five vertices and ten arcs joining these vertices in pairs so that the arcs are disjoint except for their endpoints. This is the 1-skeleton of a 4-simplex.

The principal theorem of the paper asserts that $M\in\mathfrak{M}$ is the universal curve if and only if M contains no open subset imbeddable in the plane. From this theorem it follows that $M\in\mathfrak{M}$ is the universal curve if and only if every open subset of M contains a primitive skew curve, or even a primitive skew curve of designated type. In addition to this characterization, the principal theorem also has the following corollaries. Any continuum which is the sum of a finite number of universal curves such that the intersection of any pair contains no isolated point of itself is a universal curve. Let $K\in\mathfrak{K}$ and $\varepsilon>0$ be given. If, for each connected open set D in K with diameter less than ε and for any pair of points $p, q\in D$, there is a universal curve or even a primitive skew curve in D containing p and q, then K is the universal curve. The author shows in the paper reviewed above that the universal curve is homogeneous. He now shows that the simple closed curve and the universal curve are the only homogeneous elements of $\mathfrak{K}$ and that every connected open subset of the universal curve is n-point homogeneous. Finally, it is shown that a necessary and sufficient condition that a 1-dimensional, locally compact, noncompact, connected and locally connected metric space be homogeneous is that it be homeomorphic to a component of a set which is the difference of a homogeneous element $K\in\mathfrak{K}$ and a totally disconnected closed subset of K. *W. R. Utz* (Columbia, Mo.)

2677a:

Neugebauer, Christoph J. Local A-sets, B-sets, and retractions. Illinois J. Math. 2 (1958), 386–395.

2677b:

Neugebauer, Christoph J. A fine-cyclic additivity theorem for a functional. Illinois J. Math. 2 (1958), 396–401.

The author continues his study of B-sets and fine-

cyclic elements of a Peano space of finite degree of multicoherence $r(P)<+\infty$. As in his previous paper [C. J. Neugebauer, Trans. Amer. Math. Soc. **88** (1958), 121–136; MR **20** #1959] a subset B of a Peano space is said to be a B-set provided B is a nondegenerate continuum and either $B=P$, or every component of $P-B$ has only a finite number of frontier points. A subset Γ of P is said to be a fine-cyclic element provided Γ is a B-set and no finite set of points disconnects Γ. The present papers show that B-sets and fine-cyclic elements possess properties completely analogous to A-sets and proper cyclic elements. For $r(P)=0$ fine-cyclic elements and Whyburn's cyclic elements coincide, while for $r(P)<+\infty$ fine-cyclic elements represent a finer decomposition of P. The concept of fine-cyclic element, which had been introduced by the reviewer [Riv. Mat. Univ. Parma **7** (1956), 149–185; MR **19**, 1168; and #2419 above] in surface area theory in terms of plane topology, has here a general treatment in terms of analytic topology for Peano spaces.

The first part of the first paper centers around the following main theorem: I. If $r(P)<+\infty$, then there is a finite set $B_1, \cdots, B_n$ of B sets of P such that $P=B_1\cup\cdots\cup B_n$, $B_i\cap B_j$ is either empty or finite, $i\neq j$, $i, j=1 \cdots, n$, and each proper fine-cyclic element of P is a proper cyclic element of a unique B_i, $1\leqq i\leqq n$.

In the second part of the first paper the author terms a subset B of P a local A-set of P provided either $P=B$, or there is an open connected set G of P, $B\subset G\subset P$, such that if $\{O\}$ is the collection of all components of $G-B$, then (i) for every $O\in\{O\}$ the boundary of O in G is a single point, (ii) if O', $O''\in\{O\}$ have distinct boundary sets in G, then O', O'' have disjoint closures. B is then also called a (G, A)-set of P. II. If B is a (G, A)-set of P, then B is a B-set of P. Every B-set of P is a (G, A)-set of P.

The usual concept of retraction is extended as follows: if B is a (G, A)-set of P, then a continuous mapping t from P onto B is said to be a retraction provided $t|G$ is the usual retraction from G onto B and $t(P-G)$ is the subset of a dendrite $E\subset B$. III. If B is a (G, A)-set of P, then there exists a retraction t from P onto B.

In the second paper the author extends to Peano spaces, unrestricted factorizations and functionals, the fine-cyclic additivity theorem proved by Cesari for area as well as the cyclic additivity theorem proved by C. B. Morrey, T. Rado, E. J. Mickle. IV. If $r(P)<+\infty$, if $T:P\to P^*$ is a continuous mapping from P into a metric space P^*, if $T=sf$ is an unrestricted factorization of T, $f:P\to M$; $s:M\to P^*$, if $\{\Delta\}$ is the (countable) collection of all fine-cyclic elements of M, and, for each Δ, t_Δ denotes the retraction of G_Δ onto Δ (Δ being a (G_Δ, Δ)-set of M), and $A_\Delta=f^{-1}(G_\Delta)$, then we have $\Phi(T, P)=\sum\Phi(st_\Delta f, A_\Delta)$, where $\sum$ ranges on $\{\Delta\}$. The functional Φ is supposed to be non-negative, lower semi-continuous, zero if M is a dendrite, and additive "under partition", i.e., for partitions of P into two parts whose images have finite intersection. *L. Cesari* (Baltimore, Md.)

2678:

Mrówka, S. On local topological properties. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 951–956, LXXX. (Russian summary)

Let **P** be a class (a property) of topological spaces; then **P** is called additive if $X=A\cup B$, A, B closed in X, $A\in\mathbf{P}$, $B\in\mathbf{P}$ implies $X\in\mathbf{P}$. If X is such that every $x\in X$ has a neighborhood U with $\bar{U}\in\mathbf{P}$, then $X\in\operatorname{loc}\mathbf{P}$ is written. It is proved that (1) normality and paracompactness are additive; (2) if **P** satisfies (W) "if there is a point $x\in X$ possessing arbitrarily small neighborhoods U with $X-U\in\mathbf{P}$, then $X\in\mathbf{P}$", then every $X\in\operatorname{loc}\mathbf{P}$ is contained as an open set in some $Y\in\mathbf{P}$; (3) normality and paracompactness satisfy (W); hence a locally normal (paracompact) space may be imbedded as an open set into a normal (paracompact) space [cf. E. Čech, Ann. of Math. **33** (1937), 823–844]. {Reviewer's remark: as for (1), it is well known [cf. e.g. M. Katětov, Czechoslovak Math. J. **2**(77) (1952), 333–368; MR **15**, 815] that if $X=\bigcup A_\nu$, A_ν closed normal, $\{A_\nu\}$ locally finite, then X is normal; it is easy to see that this proposition holds also with "paracompact" instead of "normal".} *M. Katětov* (Prague)

2679:

Levšenko, B. T. On the concept of compactness and point-finite coverings. Mat. Sb. N.S. **42**(84) (1957), 479–484. (Russian)

The principal results are Theorem 2: a regular space is countably compact if and only if every point-finite open covering has a finite subcovering; and Theorem 3, which asserts the equivalence of four properties for a regular space R. The first is that R is a metrizable space which is the union of a compact and a discrete subset; the others are that the family of all point-finite, locally-finite, resp. star-finite open coverings, ordered by refinement, has a countable cofinal subset. The author gives proofs of two related results, both credited to Yu. M. Smirnov; the first (a countably compact metacompact space is compact) is due to Arens and Dugundji [Portugal. Math. **9** (1950), 141–143; MR **12**, 434].

J. Isbell (Seattle, Wash.)

2680:

Kerstan, Johannes. Zur Definition der bikompakten Räume. Math. Nachr. **17** (1958), 19–21.

A covering of a space has local cardinal $<\aleph_\alpha$ if each point is contained in $<\aleph_\alpha$ sets. Theorem: If each open covering of a space contains a subcovering of local cardinal $<\aleph_\alpha$, then each open covering contains a subcovering of cardinal $<\aleph_\alpha$. In particular, a space is bicompact (Lindelöf) if and only if each open covering contains a subcovering with local cardinal $<\aleph_0$ ($<\aleph_1$). A lattice-theoretic analog, applicable to certain classes of (not necessarily open) coverings, is also proved.

J. Dugundji (Los Angeles, Calif.)

2681:

Iséki, Kiyoshi. A theorem on continuous convergence. Proc. Japan Acad. 33 (1957), 355–356.

The author generalizes a theorem of Sierpiński [General topology, Univ. of Toronto Press, 1952, p. 156; MR **14**, 394] to non-metric spaces. Specifically, it is shown that in a weak separable completely regular space S, the three conditions — (1) S is sequentially compact, (2) S is countably compact, (3) for sequences of real-valued functions, continuous convergence on S implies uniform convergence — are equivalent. *J. Dugundji* (Los Angeles, Calif.)

2682:

Iséki, Kiyoshi. Pseudo-compactness and strictly continuous convergence. Proc. Japan Acad. 33 (1957), 424–428.

The author defines a sequence $\{f_n\}$ of (real-valued) functions on a space to be strictly continuously convergent to f if, whenever $\{f(x_n)\}$ converges, $\{f_n(x_n)\}$ converges to the same limit. He proves that, in a completely regular space S, the following three conditions are equivalent:

(1) S is pseudo-compact; (2) every sequence $\{f_n\}$ of continuous functions on S convergent strictly continuously to a continuous f converges uniformly to f on S; (3) every equicontinuous sequence of functions converging pointwise to a continuous function converges uniformly to it. *J. Dugundji* (Los Angeles, Calif.)

2683a:

Iséki, Kiyoshi. On generalized continuous convergence. Proc. Japan Acad. **33** (1957), 525–527.

2683b:

Iséki, Kiyoshi. New characterisations of compact spaces. Proc. Japan Acad. **34** (1958), 144–145.

The author has announced [#2681, #2682 above] characterizations of pseudo-compact spaces in terms of convergence of sequences of functions. The conclusion from the present additions and corrections is that these and similar characterizations are valid in the case of countably paracompact normal spaces.

J. Isbell (Seattle, Wash.)

2684:

Iséki, Kiyoshi. A remark on pseudo-compact spaces. Proc. Japan Acad. 33 (1957), 528–529.

Another characterization of pseudo-compactness in terms of convergence of sequences of functions.

J. Isbell (Seattle, Wash.)

2685:

Isiwata, Takesi. On strictly continuous convergence of continuous functions. Proc. Japan Acad. **34** (1958), 82–86.

Let X be a topological space and $C(X)$ the space of real continuous functions on X. A topology for $C(X)$ is admissible if $(f, x)\to f(x)$ is a continuous mapping of $C(X)\times X$ into the real numbers. A sequence $\{f_n\}$ of functions in $C(X)$ converges jointly to $f\in C(X)$ [$f_n\to f$ (jointly)] if $f_n\to f$ in some admissible topology for $C(X)$. $\{f_n\}$ converges continuously to f [$f_n\to f$ (cont.)] if $x_n\to x$ implies $f_n(x_n)\to f(x)$. $\{f_n\}$ converges strictly continuously to f [$f_n\to f$ (str. cont.)] if $f(x_n)\to\alpha$ implies $f_n(x_n)\to\alpha$. If continuous convergence implies strict continuous convergence, then X is said to have property S. Theorem 1: Let X be countably compact and T_1. Then $f_n\to f$ (jointly)$\Leftrightarrow f_n\to f$ (str. cont.)$\Leftrightarrow f_n\to f$ uniformly. Theorem 2: The property S is preserved under passage from a space to a dense subspace. Theorem 3: Let X be infinite and discrete and βX the Stone-Čech compactification of X. Then βX has no convergent sequence, so that every sequence $\{f_n\}$ converges continuously to every f. However there is a sequence $\{f_n\}$ that does not converge strictly continuously to zero. [See also #2682 above]. *E. Hewitt* (Seattle, Wash.)

2686:

Isiwata, Takesi. The space of measures on a countably compact T_1-space. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A **5** (1957), 295–299.

Let X be a topological space, let M be the set of all normalized regular outer measures, μ, satisfying $\mu(U)+\mu(X-U)=1$ for each open subset U of X and let M be endowed with the weak topology. The author considers the following proposition: If X is a countably compact T_1-space then M is a compact Hausdorff space. A proof of this theorem will answer a question posed by J. H. Blau [Fund. Math. **38** (1951), 23–34; MR **13**, 830].

The technique employed by the author is to map M homeomorphically into a certain compact Hausdorff space $\mathfrak{X}$. The compactness of M will then follow if $f(M)$ can be shown to be a closed subset of $\mathfrak{X}$. Now (from the definition of $\mathfrak{X}$) any limit point of $f(M)$ can be thought of as a set function defined on the open subsets of X. The author shows, without difficulty, that the extension to all subsets of X given by $\mu(A)=\inf\{\mu(U):A\subset U,\ U \text{ is open}\}$ is a normalized, regular outer measure. The proof will be complete if the open sets can be shown to be members of the Caratheodory σ-ring determined by this outer measure. However, at this point there seems to be a gap in the author's argument. The paper also contains several corollaries of the above theorem.

R. E. Zink (Lafayette, Ind.)

2687:

Heider, L. J. A note concerning completely regular G_δ-spaces. Proc. Amer. Math. Soc. **8** (1957), 1060–1066.

Let $C(X)$ denote the set of all continuous functions on a completely regular space X. A dense subspace Y of X such that every $f\in C(Y)$ has a continuous extension over X is called an imbedded subspace of X. A generalized G_δ-point p of X is a point p of X such that $X-\{p\}$ is not an imbedded subspace of X. In this paper the author describes the nature of the imbedded subspaces of vX [for definition, see Hewitt, Trans. Amer. Math. Soc. **64** (1948), 45–99; MR **10**, 126], and gives some instructive examples. His main results are the following.

The set μX of generalized G_δ-points of X is the intersection of all the imbedded subspaces of vX, but μX need not be an imbedded subspace of vX. [See also the reviewer's paper in Michigan Math. J. **4** (1957), 61–64; MR **18**, 916.] If $X=\mu X$, and $Y=\mu Y$, then a ring, lattice, or semigroup isomorphism of $C(X)$ onto $C(Y)$ determines a homeomorphism of X onto Y. A generalized G_δ-point p of X such that every bounded $f\in C(X-\{p\})$ has a continuous extension over X is a G_δ-point of X.

M. Henriksen (Lafayette, Ind.)

2688:

Henriksen, Melvin; and Isbell, J. R. Local connectedness in the Stone-Čech compactification. Illinois J. Math. **1** (1957), 574–582.

This paper continues the investigation undertaken by A. D. Wallace [Trans. Amer. Math. Soc. **70** (1951), 97–102; MR **12**, 845] and Banaschewski [Canad. J. Math. **8** (1956), 395–398; MR **17**, 1229] of local connectedness in the Stone-Čech compactification βX of a completely regular space X. Banaschewski showed, essentially, that if βX is locally connected then X is locally connected and pseudo-compact (i.e., every continuous real function on X is bounded); and Wallace, that when X is normal, βX is locally connected if and only if every finite open covering of X has a finite refinement consisting of connected sets. Here, a different proof of Banaschewski's theorem is given, and the converse obtained. Moreover, it is shown that Wallace's condition on X is equivalent, for any regular X, to local connectedness and countable compactness, so that Wallace's theorem is seen to be a special case. It is also found that if βX is locally connected then, in fact, every completely regular space in which X is dense is locally connected.

Some pointwise statements are also obtained: βX is not locally connected at any point not in the subspace $\langle aX\rangle$, the completion of X in its finest uniformity; βX is locally connected at a point of X if and only if X is locally connected there; and if X is locally connected, then βX is locally connected at every point of $\langle aX\rangle$. Several other facts of interest are presented in the course of the development. An open set $U\subset\beta X$ is connected if and only if $U\cap X$ is connected. (This is contained implicitly in

Banaschewski's work, and generalizes a lemma of Wallace for normal spaces.) A space X is locally connected if and only if every normal covering has a normal refinement consisting of connected sets. Finally, X is locally connected if and only if $\langle aX \rangle$ is locally connected.

{There are several misprints and minor misstatements. For instance, in the proof of 1.4, "$V_1^\beta \cap V_2^\beta$" should be "$V_1^\beta \cap V_2^\beta \cap U$"; and in the proof of 2.2, "$\bar{V}_{n\gamma}$" should be "$\bar{V}_{n\gamma} \cap X$".} *C. W. Kohls* (Urbana, Ill.)

2689:

Henriksen, Melvin; and Isbell, J. R. Some properties of compactifications. Duke Math. J. **25** (1957), 83–105.

This paper studies the properties of sets of points which may be added to a Hausdorff space X in compactifying it. Since only completely regular spaces have compactifications, the paper deals (with the exception of the appendix) only with such spaces. (Hence the term space, except in the appendix, is used to mean a completely regular space.) A fundamental tool in this work is the well-known result of Čech: Any compactification BX of X is the image of βX under a continuous mapping f such that $f(x)=x$ for $x \in X$ and $f(\beta X - X) = BX - X$.

The following definitions are introduced: A closed continuous mapping f of X onto Y such that, for each $y \in Y$, the set $f^{-1}(y)$ is compact is called a fitting map. A continuous mapping of X onto Y such that there exist compactifications AX of X and BY of Y and a continuous extension $\bar{f}$ of f over AX onto BY such that f maps $AX-X$ homeomorphically onto $BY-Y$ is called a meshing map. Evidently every meshing map is a fitting map.

In § 1, the authors study the above-defined mappings. It is shown, among other things, that there are many fitting maps that are not meshing. § 2 is devoted to the study of fitting and meshing properties: A property $\mathscr{P}$ of a topological space is called a fitting (meshing) property if, whenever f is a fitting (meshing) map of X onto Y, X has property $\mathscr{P}$ if and only if Y has property $\mathscr{P}$. Since every meshing map is a fitting map, it is clear that every fitting property is a meshing property. An example of a meshing property that is not fitting is given. A basic result of this section is Theorem 2.2, which states: The following properties are fitting properties: compactness, σ-compactness, the Lindelöf property, countable compactness, local compactness, paracompactness, and countable paracompactness. (Parts of Theorem 2.2 for normal X and Y are already given by Hanai [Proc. Japan Acad. **32** (1956), 388–391; MR **18**, 225]; more fitting properties are also deducible from that work.) The properties $\mathscr{P}$ such that for all spaces X "(*) if the complement of X in one of its compactifications has property $\mathscr{P}$, then the complement of X in any of its compactifications has property $\mathscr{P}$" are studied. It is shown that a necessary and sufficient condition for (*) to hold for a property $\mathscr{P}$ is that it be a meshing property. Thus for the properties listed in Theorem 2.2, (*) holds. The authors observe that "the essence of the reason that (*) holds for the listed properties is that the restriction of [the Čech mapping given above] f to $\beta X - X$ preserves these properties in the strong sense that the domain has the property if and only if the range does." X is said to have property $\mathscr{P}$ at infinity if $\beta X - X$ has property $\mathscr{P}$. Other results of this section include: A property $\mathscr{P}$ is fitting if and only if $\mathscr{P}$ at infinity is. If $\mathscr{P}$ is a meshing property, then $\mathscr{P}$ at infinity is also a meshing property.

In § 3 special properties at infinity are discussed. Compactness at infinity is equivalent to local compactness. Results here include: X is locally compact at infinity if and only if the set of all points of X at which X is not locally compact is compact. X is Lindelöf at infinity if and only if every compact subset of X is contained in a compact set of countable character. (As a corollary of this, every metrizable space is Lindelöf at infinity).

In § 4, the appendix, it is shown that for Hausdorff spaces complete regularity is not a fitting property, while regularity is. *Hing Tong* (Middletown, Conn.)

2690:

Yarutkin, N. G. On generalized proximity spaces. Mat. Sb. N.S. **43(85)** (1957), 397–400. (Russian)

The author establishes the existence of an absolutely closed extension using still weaker separation axioms than in his earlier paper [Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki. **10** (1956), 61–79; MR **19**, 436]. Using a stronger separation axiom, he establishes uniqueness of the structure of a compact space. *J. Isbell* (Seattle, Wash.)

2691:

Robertson, A. P.; and Robertson, Wendy. A note on the completion of a uniform space. J. London Math. Soc. 33 (1958), 181–185.

A uniformly continuous mapping is said to satisfy the filter condition if the image of every non-convergent Cauchy filter is non-convergent. For a one-to-one mapping to have a one-to-one extension over the completion, the filter condition is necessary but not sufficient. Suppose G and H are topological groups having group completions G^-, H^-, f a continuous homomorphism $G \to H$, and E a subset of G such that f is one-to-one on E and satisfies the filter condition on EE^{-1}; then the extension of f over G^- is one-to-one on the closure of E.

{Reviewer's remark: The filter condition is equivalent to the inverse images of complete sets being complete, in locally complete spaces; thus these mappings generalize proper mappings.} *J. Isbell* (Seattle, Wash.)

2692:

Nagata, Jun-iti. On imbedding a metric space in a product of one-dimensional spaces. Proc. Japan Acad. 33 (1957), 445–449.

The author has earlier shown [Proc. Japan Acad. **32** (1956), 568–573; MR **19**, 300] that every n-dimensional metric space can be imbedded in a product of $n+1$ one-dimensional spaces. In the present paper he proves that every metric space R can be topologically imbedded in a product of an enumerable number of functional spaces R_i with $\dim R_i \leq 1$ $(i=1, 2, \cdots)$.

W. W. S. Claytor (Washington, D.C.)

2693:

Nagata, Jun-iti. Note on a theorem for metrizability. Proc. Japan Acad. 33 (1957), 613–615.

The following metrization theorem is proved, using another metrization theorem recently obtained by the author [Proc. Japan Acad. 33 (1957), 128–130; MR **19**, 157]. Theorem: A T_1-space X is metrizable if and only if each $p \in X$ has a neighborhood basis $\{V_n(p) \mid n=1, 2, \cdots\}$ such that (a) for every $p \in X$ and n there exists $m = \alpha(p, n)$ such that $p \in V_m(q)$ implies $V_m(q) \subset V_n(p)$, (b) for every $p \in X$ and n there exists $l = \beta(p, n)$ such that $q \in V_l(p)$ implies $p \in V_n(q)$. Corollary: A T_1-space X is metrizable if and only if one can assign to each $p \in X$ a neighborhood basis $\{V_n(p) \mid n=1, 2, \cdots\}$ such that, for every $A \subset X$, $\bigcap_{n=1}^{\infty} \bigcup_{p \in A} V_n(p) = \bar{A}$.

E. Michael (Seattle, Wash.)

2694:

Atuji, Masahiko. Solution of problem 6.2.16 (in the case of a distance space). Sûgaku 8 (1956/57), 152–153. (Japanese)

The author considers the following question: In what case has a uniform space with a given uniform structure the property stated in the problem? He restricts himself to considering distance spaces with the ordinary uniform structure defined by $V_{1/n}=\{(x, y); d(x, y)<1/n\}$, where $d(x, y)$ is the distance (real function) between two points x and y. Let us assume that the real number space has also a uniform structure of the same kind. The terminology used in the present paper is that of Bourbaki [Topologie générale, Chap. IX, Hermann, Paris, 1948; MR **10**, 260]. We write $\alpha(x)=\{y; d(x, y)<\alpha\}$ for a positive real number α and $r(x)=\sup\{\alpha; \alpha(x)=x, \alpha>0\}$ for an isolated point x. The following theorems are proved. (1) In order that, in a distance space, a continuous real function constructed from $A=\{\alpha_n(x_n)\}$ always be uniformly continuous, it is necessary and sufficient that, if $x_m\neq x_n$, $m\neq n$ and $\{x_n\}$ has no accumulation point, x_n be isolated points, except for a finite number of points, and $\inf r(x_n)>0$. (2) The necessary and sufficient condition for any continuous real function in a distance space to be uniformly continuous is that any bounded continuous function be always uniformly continuous. *A. Kawaguchi* (Sapporo)

2695:

Hamstrom, Mary-Elizabeth; and Dyer, Eldon. Regular mappings and the space of homeomorphisms on a 2-manifold. Duke Math. J. **25** (1958), 521–531.

Using different techniques the authors first obtain extensions to arbitrary compact manifolds of Kneser's result concerned with the space of rotations of the 2-sphere as a subset of the space of orientation-preserving homeomorphisms of the sphere onto itself. It is shown that for an arbitrary compact manifold M with boundary $B(M)$ and any closed subset s of $B(M)$ with only a finite number of components, the space of orientation-preserving homeomorphisms of M onto itself leaving s pointwise fixed is locally contractible. Also the space of homeomorphisms of an annulus onto itself which leave one of the boundary curves pointwise fixed is contractible. Extensions are also obtained of earlier results of the authors on regular mappings. If Y meets certain conditions, it is shown that if f is a 0-regular mapping of a metric space X onto Y such that each point inverse under f is homeomorphic with a given compact 2-manifold, then X is a locally trivial fibre space with projection map corresponding to f. *G. T. Whyburn* (Charlottesville, Va.)

2696:

Borisovič, Yu. G. On the genus of sets. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. **1957**, no. 5, 3–7. (Russian)

A new generalization of the genus of a set is given [cf. Krasnosel'skiĭ, Uspehi Mat. Nauk (N.S.) **7** (1952), no. 2 (48), 157–164; Borisovič, ibid. **12** (1957), no. 1(73), 157–160; Dokl. Akad. Nauk SSSR **104** (1955), 165–168; MR **14**, 55; **19**, 755, 564]. Let R be a complete metric space, and $A_1, \cdots, A_m$ continuous mappings of R into R; a closed subset E is said to be of genus 1 with respect to the mappings $A_1, \cdots, A_m$, symbolically $r[E; A_1, \cdots, A_m]=1$, provided no connected component of E contains a set of the form $\{x, A_1x, \cdots, A_mx\}$ [for the case $A_i=A^i$, $i=1, \cdots, m-1$, $A^m=I$, cf. Krasnosel'skiĭ, loc. cit.]. There lation $r[E; A_1, \cdots, A_m]=n$ is defined as in the case $m=1$. The statements in connexion with his generalized r are like those for $m=1$. In particular, for a compact E containing no fixed points of transformations $A_1, \cdots, A_m$, one has $r[E; A_1, \cdots, A_m]\leq\dim E+1$ (analogue of a Lyusternik-Šnirel'man relation concerning the category); if, moreover, r is finite, there exists an $\varepsilon>0$ such that the sphere $S(E, \varepsilon)$ and E have the same genus. *Đ. Kurepa* (Zagreb)

2697:

★Alexandroff, P. [Aleksandrov, P.S.] Aus der mengentheoretischen Topologie der letzten zwanzig Jahren. Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, pp. 177–196. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 582 pp. $7.00.

Expository lecture, treating, in a descriptive fashion, three topics: (I) metrization; (II) dimension; (III) duality; with some historical remarks. Part I: The Nagata-Smirnov metrization theorem. Efremovich's neighborhood spaces and their connection with uniform spaces. Aleksandrov's version of Čech-compactification, and Smirnov's theorem on the 1-1 correspondence between compactification of the (completely regular) space R, and the neighborhood space structures on R. Part II: Results of Pontryagin, Bokšteĭn and Boltyanskiĭ on dimension of product spaces. Aleksandrov's dimensional obstruction theorem and Sitnikov's sack-and-girdle theorem [K. Sitnikov, Dokl. Akad. Nauk SSSR (N.S.) **66** (1949), 1059–1062; **81** (1951), 153–156; MR **11**, 45; **13**, 487]. Open problem: For compact φCE^n let $\alpha^p\varphi$ be the lower bound of the ε for which there is an ε-shift of φ into a p-dimensional polyhedron. Let φ be finite union of n-simplices, φ^* its boundary, $p<n-1$; is $\alpha^p\varphi=\alpha^p\varphi^*$? This is open even for $n=3$, φ homeomorphic to a solid torus. Part III is a survey of the duality theorems of Aleksandrov and Sitnikov [P. Aleksandrov, Mat. Sb. N.S. **21** (1947), 161–232; **33** (1953), 241–260; MR **9**, 456; **16**, 503; Sitnikov, ibid. **34** (1954), 3–54; MR **16**, 736]. *H. Samelson* (Ann Arbor, Mich.)

ALGEBRAIC TOPOLOGY

See also 2374, 2392, 2647, 2648, 2697.

2698:

Matschinski, Matthias. Sur la classification des polytopes saturés. C. R. Acad. Sci. Paris **245** (1957), 2461–2464.

A "saturated polygon" is an N-gon dissected by $N-3$ diagonals into $N-2$ triangles. The author gives formulae for the numerical properties of the analogous "saturated polytope" which is a simplicial n-complex whose boundary is an $(n-1)$-sphere containing all the cells of dimension $<n-1$. *H. S. M. Coxeter* (Toronto, Ont.)

2699:

Sieklucki, K. On a contractible polytope which cannot be metrized in the strong convex manner. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 361–364.

This "polytope" is a 2-complex B, derived from a disk by identifying a radius with the circumference. Borsuk [Fund. Math. **37** (1950), 137–160; MR **13**, 55] proved that B is contractible to a point, and conjectured that it cannot be metrized in the strong convex manner (a metric is said to be "strong convex" if it provides a unique midpoint

between any two given points). The author establishes the truth of this conjecture.

H. S. M. Coxeter (Toronto, Ont.)

2700:

Davis, Chandler. Another subdivision which can not be shelled. Proc. Amer. Math. Soc. **9** (1958), 735–737.

A subdivision of a 3-cell consisting of 3-cells $B_1, \cdots, B_k$ cannot be shelled if $\bigcup[B|j\neq i]$ is not a 3-cell for each i. The author constructs two such subdivisions consisting of only three pieces. These constructions are then generalized to prove the following two theorems.

1. For $k\geqq n\geqq 3$, there exists a subdivision of the n-cell into k pieces such that the union of any $k-1$ of the pieces has nontrivial $(n-2)$th homotopy group.

2. For any $n=1, 2, 3, \cdots$, there exists a subdivision of the 3-cell into three pieces such that omitting any of the pieces leaves a set whose fundamental group is the free group on n generators.

R. Ellis (Philadelphia, Pa.)

2701:

Čogošvili, G. S. On Čech groups of infinite chains and finite cochains. Soobšč. Akad. Nauk Gruzin. SSR. **19** (1957), no. 5, 513–520. (Russian)

The author considers the set S of all open star-finite coverings O_α of a topological space X. S is ordered by $<$, where $O_\alpha<O_\beta$ means that O_β refines O_α and that every element of O_α contains at most finitely many elements of O_β. There are no difficulties in considering infinite chains of nerves of O_α (coefficients in a compact group C), finite cochains (coefficients in a discrete group D), boundary and coboundary operators and mappings induced by $<$. However, $(S, <)$ may consist of more than one maximal directed subset τ. Taking corresponding limits of homology and cohomology groups for each τ separately, the author arrives at a collection of homology and cohomology groups $\{H_{\Delta\tau}{}^r(X; C)\}$ and $\{H_{\nabla\tau}{}^r(X; D)\}$ that he calls the rth homology [cohomology] aggregate of X, and which replace the usual Čech groups.

Corresponding definitions are given for relative groups, for induced homomorphisms and boundary homomorphisms. A form of the Eilenberg-Steenrod axioms is verified (the homotopy axiom is not discussed), and topological invariance is established. The exactness axiom is fulfilled for arbitrary pairs (X, A) in both homology (coefficients in C) and cohomology.

S. Mardešić (Princeton, N.J.)

2702:

Zeeman, E. C. On the filtered differential group. Ann. of Math. (2) **66** (1957), 557–585.

An abelian group A with differentiation d: $A\to A$, $dd=0$, and a filtration $0=A_0\subset A_1\subset\cdots\subset A_m=A$ is considered. With such a system $\mathfrak{A}=\{A, A_p, d\}$ there is associated a category $\mathfrak{A}^\#$ of groups and homomorphisms suitably generated by d and the inclusion maps of the A_p's and closed under the operations on groups and homomorphisms that one may reasonably wish to perform. The rather vague assumption is that the category $\mathfrak{A}^\#$ is actually independent of $\mathfrak{A}$. The terms of the spectral sequence of $\mathfrak{A}$ or of the exact couple of $\mathfrak{A}$ all appear in the category $\mathfrak{A}^\#$. The author succeeds in solving what is essentially the analogue of the "word problem" in the category $\mathfrak{A}^\#$. This is achieved by a mapping λ of $\mathfrak{A}^\#$ onto a category whose objects are regions made up of squares in the plane. The essential result is (Th. 4) that λ is in a suitable sense faithful and most efficient. In this way the manipulation of various groups in $\mathfrak{A}^\#$ reduces to operations on regions. This is illustrated by the example of fiber bundles. Finally, the difficulties inherent in a similar treatment of bi-filtered differential groups are discussed. It appears that the main difficulty may be traced to the non-distributivity of a certain lattice which in the case of a (singly) graded differential group is distributive.

S. Eilenberg (New York, N.Y.)

2703:

Kan, Daniel M. On the homotopy relation for c.s.s. maps. Bol. Soc. Mat. Mexicana **2** (1957), 75–81.

Let C be any category and $\mathscr{V}$ the category consisting of the "simplexes" of semisimplicial theory and the c.s.s. maps between them; then $C^{\mathscr{V}}$ denotes the category whose objects are contravariant functors $K:\mathscr{V}\to C$ and whose maps are natural transformations of such functors. $C^{\mathscr{V}}$ is called the c.s.s. category over C. If C allows a notion of sum (generalising the idea of union in the category of sets), then in $C^{\mathscr{V}}$ a notion of homotopy can be defined. The main theorem of the paper states a simple sufficient condition under which a covariant functor $\theta:C^{\mathscr{V}}\to D^{\mathscr{V}}$ preserves homotopies; roughly speaking, the condition is that for A an object of $C^{\mathscr{V}}$, the n-dimensional part of θA should depend on the n-dimensional part of A only.

V. Gugenheim (Baltimore, Md.)

2704:

Kan, Daniel M. On c.s.s. categories. Bol. Soc. Mat. Mexicana **2** (1957), 82–94.

This paper continues the study of c.s.s. categories (see preceding review); homotopy-groups, the extension condition and the notion of a fibre-space are generalized to this context, and various homotopy-extension and covering theorems are proved; as well as an analogue of the theorem of J. H. C. Whitehead which asserts that a map inducing isomorphism of all homotopy-groups is a homotopy-equivalence. The main tool in these investigations is a functor from the c.s.s. category $C^{\mathscr{V}}$ to the usual category of c.s.s. complexes, which allows one to transfer most of the old definitions to the new context.

V. Gugenheim (Baltimore, Md.)

2705:

Shih, Weishu. Sur la suite exacte d'homotopie. C. R. Acad. Sci. Paris **246** (1958), 2833–2835.

Let X, Y be Kan complexes and f: $(X, x_0)\to(Y, y_0)$ simplicial, where x_0, y_0 are vertices. Define $[Y, X]^f$ to be the (Kan) complex with: (1) n-simplexes: all pairs (y, x) consisting of an $(n+1)$ simplex $y\in Y$ and an n-simplex $x\in X$ which satisfy $\partial_0 y=f(x)$; and (2) face and degeneracy operators: $\partial_i=(\partial_{i+1}, \partial_i)$, $s_i=(s_{i+1}, s_i)$. The projection $(y, x)\to x$ exhibits $[Y, X]^f$ as a Kan fiber space over X, and the fiber over x_0 can be identified with the Moore complex Y^* of the loop space of Y based at y_0 [J. C. Moore, Sém. H. Cartan, 1954/55, Paris, 1955; MR **19**, 438; exposé 19]. Defining $\pi_n{}^f=\pi_n[Y, X]^f$ and identifying $\pi_{n+1}(Y)$ with $\pi_n(Y^*)$, the homotopy sequence of this fibering leads to the exact sequence

$$\cdots\to\pi_{n+1}(X)\to\pi_{n+1}(Y)\to\pi_n{}^f\to\pi_n(X)\to\cdots.$$

The following are special cases of this construction and exact sequence. (a) If f is an injection, or a fiber map, the exact sequence reduces to the usual one. (b) If C is the functor sending each Kan complex to its group of G-chains, then by using $H_n(Y; G)\equiv\pi_n(C(Y; G))$, the above exact sequence for $C(f)$ is the usual homology sequence of f. (c) If i: $X\to C(X)$ sends each simplex x to the chain $1\cdot x$, the above exact sequence for i is the Whitehead exact sequence. The author shows that $\pi_n{}^f$ is an invariant of the

homotopy type of the triple (X, Y, f), and determines the class of $\pi_n{}^f$ in the group $\mathrm{Ext}[\mathrm{Ker}\, f: \pi_n(X) \to \pi_n(Y);\ \mathrm{Coker}\, f: \pi_{n+1}(X) \to \pi_{n+1}(Y)]$.

J. Dugundji (Los Angeles, Calif.)

2706:

Shih, Weishu. Sur le système de Postnikov d'un fibré principal. C. R. Acad. Sci. Paris **246** (1958), 3145–3147.

The author determines the Postnikov decomposition of $[Y, X]^f$ [see the preceding review for the definition]. For each integer $m>0$, define an equivalence relation on X by: $a\langle m\rangle b$ if $a(m)b$ and $f(a)(m+1)f(b)$, where (m) is Moore's [Sém. H. Cartan, 1954/55, Paris, 1955; MR **19**, 438; exposé 19] mth Postnikov relation. Let $X^{\langle m\rangle}=X/\langle m\rangle$; $X^{\langle m\rangle}$ is a Kan complex, and a fiber space over Moore's mth Postnikov system $X^{(m)}$, having fiber $K(\pi, m+1)$, where $\pi=\mathrm{Im}\, f: \pi_{m+1}(X) \to \pi_{m+1}(Y)$. It is shown that the Postnikov decomposition of $[Y, X]^f$ is given by a sequence of fiberings

$$[Y,\ X]^f \to \cdots \to [Y^{(m+2)},\ X^{\langle m+1\rangle}]^f \to [Y^{(m+1)},\ X^{\langle m\rangle}]^f \to \cdots$$

As an application: The mth Postnikov system of a principal fiber space with base B and simplicial group G is a principal fiber space with group the mth Postnikov system of G, and base a fiber space over the mth Postnikov system of B having fiber of type $K(\pi, m+1)$, where π is the image of $\pi_{m+1}(B)$ in $\pi_m(G)$.

J. Dugundji (Los Angeles, Calif.)

2707:

Shih, Weishu. Sur la suite exacte d'homotopie. C. R. Acad. Sci. Paris **246** (1958), 3567–3570.

In his first paper, the author determined the group $H_n{}^{(f)} \equiv \pi_n{}^{c(f)} = \pi_n[C(Y),\ C(X)]^{c(f)}$; he now seeks the relation of this group to $H_n{}^f \equiv \pi_n(C[Y, X]^f) = H_n[Y, X]^f$. Making the simplifying assumption that all Kan complexes have a single vertex, let Ω be the fiber of the projection $C[Y, X]^f \to C(X)$ induced by $[Y, X] \to X$, and let φ be the map $C[Y, X]^f \to C(Y^{\#})$ ($Y^{\#}$=Moore's path space complex) restricted to Ω; then $\varphi: \Omega \to C(Y^*) \approx C(Y)^*$ and there results an exact sequence

$$\cdots \to H_{m+1}^{\{f\}} \to \pi_m[C(Y)^*, \Omega]^{\varphi} \to H_m^{f} \to H_m^{\{f\}} \to \cdots.$$

This reduces the problem to the determination of the intermediary group; if Y is the classifying space of a simplicial group G (which is no restriction) an algorithm for its calculation is indicated. As an application: Let E be a fiber space over X with characteristic class $\xi \in H^{n+1}(X, \pi)$. Then $H_n(E)$ is isomorphic to $H_n{}^{(\zeta)}$ where $\zeta: X \to K(\pi, n+1)$ is a map in the homotopy class corresponding to ξ; if also X is simply connected, then $H_{n+1}(E) \approx H_{n+1}{}^{(\zeta)}$.

J. Dugundji (Los Angeles, Calif.)

2708:

Shiraiwa, Kenichi. A remark on (π, n)-type CW-complexes. Nagoya Math. J. **12** (1957), 25–30.

This paper is devoted to the construction of $K(\pi, n)$-spaces which are CW-complexes, but differ from those obtained from the construction of J. H. C. Whitehead in that the number of cells is algebraically minimal to realize the integral homology group $H_*(\pi, n; Z)$.

Let π be a finitely generated Abelian group, and let n be $\geqq 2$. Decomposing $H_q(\pi, n) = F_1{}^q + \cdots + F_{r_q}{}^q + T_1{}^q + \cdots + T_{l_q}{}^q$ where the F's are infinite and the T's finite cyclic groups, the author associates a q-cell $e_i{}^q$ with each $F_i{}^q$, and a q-cell $'e_i{}^q$ and a $(q+1)$-cell $''e_i{}^{q+1}$ with each $T_i{}^q$.

Theorem. There exists a (π, n)-CW-complex

$$X = \bigcup_q \left(\bigcup_{i=1}^{r_q} e_i{}^q \bigcup_{i=1}^{l_q} {}'e_i{}^q \bigcup_{i=1}^{l_q} {}''e_i{}^{q+1} \right)$$

with $\partial e_i{}^q = \partial' e_i{}^q = 0$; $\partial'' e_i{}^{q+1} = t_i{}^q\, {}'e_i{}^q$, where $t_i{}^q$=order $T_i{}^q$.

The proof is by induction using well-known exact sequences and the Whitehead lemma about the realization of a weak homotopy equivalence of CW-complexes.

D. W. Kahn (New Haven, Conn.)

2709a:

Yoshioka, Tsunéo. L'homologie du produit cyclique d'ordre p d'un complexe fini (p premier impair). Proc. Japan Acad. **34** (1958), 32–37.

2709b:

Yoshioka, Tsunéo. Base canonique d'homologie du produit cyclique d'ordre p d'un complexe fini (p premier impair). Osaka Math. J. **10** (1958), 11–29.

In Ann. of Math. (2) **59** (1954), 570–583 [MR **15**, 890] the reviewer determined the homology of the 2-fold symmetric product of a simplicial complex K in terms of the homology of K. In the present papers the author determines the homology of the p-fold cyclic product where p is an odd prime (for $p=2$, the p-fold symmetric and p-fold cyclic products coincide). The general method used is similar to that used for $p=2$, namely the special homology of Smith-Richardson and a demonstration (by induction on dimension) that certain cycles constructed from a Künneth basis for the p-fold cartesian product constitute a basis for the homology over the integers of the cyclic product. The introduction of an auxiliary complex, $M(t)$, enables the author to localize some of the difficulties and thus simplify the argument.

S. Stein (Davis, Calif.)

2710:

Olum, Paul. Non-abelian cohomology and van Kampen's theorem. Ann. of Math. (2) **68** (1958), 658–668.

The author introduces singular cohomology groups $H^0(X, A; \Pi)$, $H^1(X, A; \Pi)$ in which the coefficient group Π is not necessarily Abelian. These groups satisfy analogues of the Eilenberg-Steenrod axioms. They are needed in order to unify the obstruction theory in a paper which the author promises us. The author also uses them to obtain a simple and conceptual treatment of that theorem of van Kampen which describes $\pi_1(A \cup B)$ in terms of $\pi_1(A)$, $\pi(_1B)$ and $\pi_1(A \cap B)$. This theorem is derived from a suitable Mayer-Vietoris equence.

J. F. Adams (Cambridge, England)

2711:

Adams, J. F. On the structure and applications of the Steenrod algebra. Comment. Math. Helv. **32** (1958), 180–214.

Let $\pi_m{}^S(X)$ denote the stable homotopy group $\lim \pi_{m+n}(S^n X)$. Let K^m be the subgroup of $\pi_m{}^S(X)$ of elements of finite order and prime to a fixed prime p. Let A denote the mod p Steenrod algebra; $H^*(X; Z_p)$ is a left module over A. The author proves the following important theorem: There is a spectral sequence such that

$$E_2{}^{s,t} \approx \mathrm{Ext}_A{}^{s,t}\,(H^*(X; Z_p),\ Z_p)$$

and E_∞ is the graded group associated with $\pi_m{}^S(X)/K^m$. He then specializes to the case where $X=S^0$. Then $\pi_m{}^S(S^0)$ is the stable m-stem of the homotopy groups of spheres, and $E_2{}^{s,t} \approx \mathrm{Ext}_A{}^{s,t}(Z_p, Z_p)$. He then makes a thorough study of the structure of A (when $p=2$) to deduce $\mathrm{Ext}_A{}^{s,t}(Z_2, Z_2)$ in case $s=0$, 1, or 2, and relations in case $s=3$. These computations are used to prove the

following theorem in homotopy theory: if $\pi_{2n-1}(S^n)$ and $\pi_{4n-1}(S^{2n})$ both contain elements of Hopf invariant one, then $n \leqq 4$. *F. P. Peterson* (Cambridge, Mass.)

2712:

Weier, Joseph. Eine Verschlingungsinvariante. Proc. Japan Acad. **34** (1958), 142–143.

Let f be a mapping of a 3-manifold into a closed orientable surface. The author defines a homotopy invariant of f called the linking type, but states no theorems. *P. A. Smith* (New York, N.Y.)

2713:

Berstein, Israël. Sur la catégorie de Lusternik-Schnirelmann. C. R. Acad. Sci. Paris **246** (1958), 362–364.

Let X be a paracompact connected locally arcwise connected topological space homotopically dominated by a CW-complex. Assume further that the cohomology groups $H^q(X, A)$ vanish for all $q > n$ and all local coefficient systems A. Let Π be the fundamental group of X and let $\dim \pi$ be the cohomological dimension of π. Finally, let $\operatorname{cat} X$ [or $\operatorname{cat}_m X$] denote the Lusternik-Schnirelmann [or the Fox m-dimensional] category of X (adjusted so that a contractible space have category zero). The author states without proof several interesting relations between these numbers:

$$\operatorname{cat} X \leqq \operatorname{cat}_m X + \left[\frac{n}{m+1}\right],$$

$$\operatorname{cat} X \leqq \max(\operatorname{cat}_{n-1} X, 1).$$

If X is a compact n-dimensional manifold V^n with

$$\pi_i(V^n) = 0 \text{ for } 1 \leqq i \leqq k, \quad k \geqq [n/2]$$

then one of the following alternatives holds:

$$n = \operatorname{cat} V^n = \operatorname{cat}_1 V^n = \dim \pi,$$

$$\operatorname{cat} V^n = \operatorname{cat}_1 V^n + 1 = \dim \pi + 1 \leqq n - k,$$

$$\dim \pi = 1, \ \operatorname{cat}_1 V^n = 2, \ \operatorname{cat} V^n = 2 \text{ or } 3.$$

The last alternative holds only if π is exceptional, i.e., $\dim \pi = 1$ and π is not free. It is not known whether such a group exists. [See Eilenberg and Ganea, Ann. of Math. (2) **65** (1957), 517–518; MR **19**, 52.]

S. Eilenberg (New York, N.Y.)

2714:

Vaccaro, Michelangelo. Gruppi fondamentali commutativi di varietà tridimensionali non orientabili. Rend. Mat. e Appl. (5) **16** (1957), 447–453.

Theorem: The only abelian groups that occur as fundamental groups of non-orientable compact 3-manifolds are Z and $Z \oplus Z_2$.

The proof is based on the consideration of the orientable double covering, and the result of Reidemeister (only Z_n, Z, and $Z \oplus Z \oplus Z$ can occur in the orientable case).

H. Samelson (Ann Arbor, Mich.)

2715:

Shimada, Nobuo. Differentiable structures on the 15-sphere and Pontrjagin classes of certain manifolds. Nagoya Math. J. **12** (1957), 59–69.

The 7-sphere bundles over S^8 are discussed; in particular, their Pontryagin classes are calculated. It is shown that the bundle-spaces which are homotopy-spheres are also topological spheres. There is one such sphere M_k for each odd integer k. It is shown that $k^2 \equiv l^2 \bmod 127$ if M_k and M_l are diffeomorphic. Hence the 15-sphere admits many distinct differential structures. All this closely follows Milnor's discussion of 3-sphere bundles over S^4 [Ann. of Math. (2) **64** (1956), 399–405; MR **18**, 498]. To each M_k corresponds an 8-sphere bundle over S^8, whose bundle-space is denoted by $\tilde{B}_k$. It is proved that $\tilde{B}_k$ and $\tilde{B}_l$ have the same homotopy type if and only if $k \equiv \pm l \bmod 240$, and this result is used to illustrate various questions about the homotopy type of manifolds and the invariance of Pontryagin classes. *I. M. James* (Oxford)

2716:

Boltyanskiĭ, V. G. Homotopy classification of vector fields. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 13–16. (Russian)

In this note the author considers smooth closed orientable manifolds M^n possessing zero Euler characteristic and gives a homotopy classification for vector fields on M^n. Let K be a smooth triangulation of M^n. If σ_1, σ_2 are two vector fields on M^n we may arrange, by making a preliminary deformation, that they coincide on K^{n-2}. Then the vector fields, restricted to K^{n-1}, determine a difference cocycle $\partial^{n-1}(\sigma_1, \sigma_2)$ whose cohomology class $D^{n-1} \in H^{n-1}(K^n, \pi_{n-1}(S^{n-1}))$ depends only on σ_1, σ_2 (and not the deformation). If we fix σ_1 we thereby associate with every vector field σ an element $D^{n-1} \in H^{n-1}(M^n)$, and it may be demonstrated (essentially by the Hopf-Whitney classification theorem) that this gives a (1, 1) correspondence between classes of fields whose restrictions to K^{n-1} are homotopically equivalent and elements of $H^{n-1}(M^n)$. It is further shown in this paper that each such class contains precisely two homotopy classes of fields if $n \geqq 4$. Indeed two vector fields σ_1, σ_2 whose restrictions to K^{n-1} are homotopic determine a difference cohomology class $D^n(\sigma_1, \sigma_2) \in H^n(K^n, \pi_n(S^{n-1}))$; this group is cyclic of order 2 and, for fixed σ_1, each value is attained.

After giving the proof of this last assertion the author closes by observing that if $n=1$ or 2 the classification problem is trivial; and if $n=3$ all manifolds M^3 under discussion are parallelizable so that the classification is achieved by the Pontryagin result classifying maps of M^3 into S^2. *P. J. Hilton* (Ithaca, N.Y.)

2717:

Tamura, Itiro. Homeomorphy classification of total spaces of sphere bundles over spheres. J. Math. Soc. Japan **10** (1958), 29–43.

In this paper the author is concerned with the topological classification of 3-sphere bundles over 4-spheres, and 7-sphere bundles over 8-spheres. As is well-known, the equivalence classes of such bundles correspond to pairs of integers so we denote them by $B_{m,n}^{(4,3)}$ and $B_{m,n}^{(8,7)}$. The author has previously computed the Pontrjagin classes of these bundles [J. Math. Soc. Japan, **9** (1957), 250–262; MR **19**, 975].

He now decomposes these spaces into curves and then produces explicit maps of some of the spaces on others by defining maps of the curves. This construction is quite complicated and very clever. These maps are then proved to be homeomorphisms. As a result one finds that $B_{m,n}^{(4,3)}$ and $B_{m',\pm n}^{(4,3)}$ are homeomorphic if $m \equiv \pm m' \bmod n$. Similarly for $B_{m,n}^{(8,7)}$.

Although the Pontrjagin classes have not been used in the construction of the homeomorphisms, the author observes that there is some connection (perhaps only by coincidence) between the Pontrjagin classes and homeomorphism. Namely, if n is odd, then spaces $B_{m,n}^{(4,3)}$ which have the same homotopy type and Pontrjagin classes, are homeomorphic.

These results are now used to improve the earlier

results on homotopy type of these spaces. We have $B_{m,n}^{(4,3)}$ and $B_{m',\pm n'}^{(4,3)}$ are of the same homotopy type if $m \equiv \pm m' \bmod (12, n)$. In particular, if $(n, 12)=1$, then $B_{m,n}^{(4,3)}$ ($m=0, \pm 1, \pm 2, \cdots$) are all of the same homotopy type.

The author next studies the question of differentiable structures on these spaces and by an extension of Milnor's methods [Ann. of Math. (2) **64** (1956), 399–405; MR **18**, 498], he shows that many of these spaces admit non-isomorphic differentiable structures.

N. Stein (New Haven, Conn.)

2718a:

Tao, Junzo. Some properties of $(n-1)$-manifolds in n-space. Proc. Japan Acad. **34** (1958), 92–95.

2718b:

Tao, Junzo. Some properties of $(n-1)$-manifolds in the Euclidean n-space. Osaka Math. J. **10** (1958), 137–146.

The second paper elaborates results which were briefly announced in the first. For a polyhedral $(n-1)$-manifold P^{n-1} in euclidean n-space R^n, locally normal position implies normal position [terminology and some methods as in S. S. Cairns, Ann. of Math. (2) **41** (1940), 796–808; **45** (1944), 218–219; MR **2**, 71; **5**, 273]. If P^{n-1} is in normal position, it admits an analytic approximation. Indeed, it is then in regular position [H. Whitney, ibid. **37** (1936), 865–878] and hence can be imbedded in a family of disjoint manifolds (analytic, save for P^{n-1} itself), homeomorphic to P^{n-1} and filling out a neighborhood thereof. For P^{n-1} compact and in regular position in R^n, the curvatura integra $d(P^{n-1})$ is defined as for a differentiable manifold, but with normal vectors replaced by a family of transversal vectors. It is shown that $d(P^{n-1})=d(Q^{n-1})$ if P^{n-1} and Q^{n-1} are isotopic; that is, correspond under an orientation-preserving semi-linear self-homeomorphism of R^n. {The condition that P^{n-1} and Q^{n-1} be isotopic was omitted from the statement of this last result (Theorem 5).} The curvatura integra of a differentiable $M^{m-1} \subset E^n$ agrees with that of an approximating P^{n-1}.

S. S. Cairns (Urbana, Ill.)

2719:

James, I. M. Embeddings of real projective spaces. Proc. Cambridge Philos. Soc. **54** (1958), 555–557.

The author points out that the problem of finding the least integer $D(m)$ such that the real projective m-space P^m can be imbedded in a Euclidean space of dimension $D(m)$ is related to a problem considered by Hopf. This and some results of Hopf lead to a formula, a special case of which is the relation

$$D(15) \leqq 2D(7)+8.$$

N. Stein (New Haven, Conn.)

2720:

Brahana, Thomas R. Products of generalized manifolds. Illinois J. Math. **2** (1958), 76–80.

Let F be a fixed field, and S a locally compact Hausdorff space. If $x \in S$, let $p_r(x)$ denote the (homology) local Betti number at x with coefficients in F. Using the global Künneth formula the author proves the result: — If S, S' are locally compact Hausdorff spaces with finite local Betti numbers at $x \in S$, $y \in S'$, then at $(x, y) \in S \times S'$

$$p_t(x, y) = \sum_{r+s=t} p_r(x) \cdot p_s(y).$$

Call S a generalised n-manifold [n-gm] if moreover $\dim S = n$ and $p_r(x)=\delta_{nr}$ [Kronecker delta]. The author then proves: If S, resp. S', is an n-gm, resp. n'-gm, such that $\dim S \times S' = n+n'$, then $S \times S'$ is an $(n+n')$-gm. A similar result is obtained when S, S' are the base and fibre of a fibre bundle.

H. B. Griffiths (Bristol)

2721:

O'Neill, B.; and Straus, E. G. A fixed point theorem. Proc. Amer. Math. Soc. **8** (1957), 1148–1151.

For topological spaces X and Y the point-to-set function $T: X \to Y$ is continuous provided the inverse function is both open and closed and $T(x)$ is closed for each $x \in X$. A mapping $f: B \to E_n$, Euclidean n-space, links the point $x \in E_n$ provided $x \notin f(B)$ and $f_*: H_{n-1}(B) \to H_{n-1}(E_n - x)$ is non-trivial, for a continuous homology theory. The authors prove the theorem: Suppose $C \subset E_n$ is compact and connected, $T: C \to C$ is a continuous point-to-set function, B is a compact space and $h_t: B \to E_n$ is a homotopy such that h_0 links every point of C and h_1 links no point of C. Then there exist a number t, $0 < t \leqq 1$, and a point $y \in C$ such that $h_t(B)$ contains both y and a point of $T(y)$. They then present two applications. (1) Let B and C be n-spheres lying in E_{n+1}, C lying in the interior of B, let $d \leqq \operatorname{diam} C$ and let h_t be a homotopy as in the theorem. Then for some t, $h_t(B) \cap C$ contains points at a distance d apart. (2) Let C be a smooth surface and B a closed surface containing C. Let B be shrunk to a point so that $B_t \cap C$ is a rectifiable curve K_t and let l be the maximum length of the K_t's. Then the minimum l for all possible contractions B_t is not less than twice the distance between nearest conjugate points on C.

E. Dyer (Chicago, Ill.)

2722a:

Liao, S. D. Periodic transformations and fixed point theorems. I. Cup products and special cohomology. Sci. Record **1** (1957), no. 1, 25–29.

2722b:

Liao, S. D. Periodic transformations and fixed point theorems. II. Manifolds. Sci. Record **1** (1957), no. 1, 31–34.

Let T be a transformation of prime period p acting on a Hausdorff space X. With the aid of certain homomorphism sequences of equivariant cohomology groups (always over Z_p), the author establishes a connection between the multiplicative structure of the cohomology of X and that of the fixed set F. In part II the author applies his methods to the case in which X is a compact manifold and shows, for example, that in this case each component of F satisfies Alexander-Poincaré duality. Further applications yield: (1) when X is a compact manifold, F cannot consist of a single point; (2) if X is an n-sphere and F a homology 2-sphere, then F is a 2-sphere; (3) if X is euclidean 4-space, then F is not empty. Only definitions and statements of results are given; details are to appear later.

P. A. Smith (New York, N.Y.)

2723:

Giorgiutti, Italo. Tableau spectral associé à une application périodique. C. R. Acad. Sci. Paris **246** (1958), 1650–1652.

Dans cette note, on considère la situation algébrique où $X = \sum X^p$ est un complexe de cochaînes sur lequel opère le groupe cyclique π d'ordre k premier engendré par un opérateur t. A l'aide d'une résolution π-projective $0 \leftarrow Z \leftarrow A_0 \leftarrow \cdots \leftarrow A_q \leftarrow \cdots$ de Z on définit le bicomplexe $C = \sum C^{p,q}$, $C^{p,q} = \operatorname{Hom}_\pi(A_q, X^p)$, d'où deux suites spectrales. L'auteur étudie la "seconde" d'entre elles au moyen du tableau spectral introduit par Deheuvels [Proc. Nat.

Acad. Sci. U.S.A. **41** (1955), 90–93; MR **17**, 520] et il obtient de la sorte des interprétations des groupes de cohomologie et des homomorphismes de Smith. A la fin de la note il aborde une application géométrique. [Voir la "review" ci-après.] *P. Dedecker* (Rhodes-St.-Genèse)

2724:

Giorgiutti, Italo. Interprétation spectrale des invariants de Smith. C. R. Acad. Sci. Paris **246** (1958), 2558–2560.

Soit un espace $\mathfrak{X}$ localement compact et paracompact muni d'une transformation t: $\mathfrak{X}\to\mathfrak{X}$ de période k. La situation algébrique de la note précédente s'applique à la cohomologie de Čech de $\mathfrak{X}$ à valeurs dans un faisceau $\mathfrak{F}$ si l'on utilise des recouvrements ouverts $\mathfrak{U}=(U_i)_{i\in I}$ dits de type (f) satisfaisant aux conditions suivantes: (a) t transforme les U_i entre eux; (b) les U_i fixes sont ceux qui rencontrent F; (c) si deux U_i rencontrent F, il en est de même de leur intersection. Ces recouvrements permettent de calculer la "bonne" cohomologie de $\mathfrak{X}$. Comme résultats, l'auteur donne une expression de $H^p(F, Z_k)$ au moyen des groupes de la théorie de Smith. Il montre que les différentielles de la suite spectrale [voir la "review" précédente] sont engendrées par les homomorphismes de Smith. Moyennant la nullité de certains groupes de cohomologie de Smith on obtient $E_\infty^{p,q}=H^p(F,\mathfrak{F})$. D'autres applications à des résultats de Floyd et Liao sont annoncées. *P. Dedecker* (Rhodes-St.-Genèse)

2725:

Conner, P. E. On the action of a finite group on $S^n\times S^n$. Ann. of Math. (2) **66** (1957), 586–588.

It is known that if a finite group acts freely on a sphere S^n then every abelian subgroup is cyclic. Using spectral sequences the author proves that if a finite group acts freely on $S^n\times S^n$ then the rank of any abelian subgroup is $\leqq 2$, and asserts that more complicated arguments would prove that if a finite group acts freely on an m-fold product $S^n\times S^n\times\cdots\times S^n$ then the rank of any abelian subgroup is $\leqq m$. In addition, it is shown that an abelian subgroup of the group of a tame knot in S^3 must be either infinite cyclic or free abelian of rank 2; this was a conjecture of the reviewer. *R. H. Fox* (Princeton, N.J.)

2726:

Aumann, Robert J. Asphericity of alternating knots. Ann. of Math. (2) **64** (1956), 374–392.

By asphericity of a knot K is meant triviality of all the higher homotopy groups of the complementary space S^3-K. The principal result of this paper — that any alternating knot is aspheric — was engulfed shortly after its publication by the far-reaching results of C. D. Papakyriakopoulos [same Ann. (2) **66** (1957), 1–26; MR **19**, 761], one consequence of which is that every knot is aspheric. The method used by Aumann is largely graph-theoretic, and seems to have no relation to the Papakyriakopoulos method. However the method is quite elegant and, as L. Neuwirth has remarked, it proves for alternating knots somewhat more than claimed; namely, that the group $\pi_1(S^3-K)$ of an alternating knot K is a free product with amalgamation $(U*V)_W$, where U and V are free groups of the same rank. *R. H. Fox* (Princeton, N.J.)

2727:

Roy, Bernard. Sur quelques propriétés des graphes fortement connexes. C. R. Acad. Sci. Paris **247** (1958), 399–401.

An oriented graph is called strongly connected if there is a path connecting any vertex to any other. The author proves that a graph Γ is strongly connected if and only if there is no set of vertices A of Γ whose successors form a proper subset of A. This is an old result first proved by R. Rado [Ann. of Math. (2) **44** (1943), 228–270; MR **5**, 151]. An algorithm is given to determine when a given graph is strongly connected. *D. Gale* (Providence, R.I.)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 2381, 2406, 2475, 2551, 2667, 2718a-b.

2728:

Strel'cov, V. V. Estimates of the length of a curve on a surface of given diameter. Akad. Nauk Kazah. SSR. Trudy Sektor. Mat. Meh. **1** (1958), 71–110. (Russian)

The paper furnishes the detailed proofs and gives some minor improvements for results announced in: A. D. Aleksandrov and V. V. Strel'cov, Dokl. Akad. Nauk SSSR (N.S.) **93** (1953), 221–224 [MR **15**, 737]. *H. Busemann* (Cambridge, Mass.)

2729:

Kaul, R. N.; and Behari, Ram. Generalization of Lie's theorem on null lines. Math. Student **25** (1957), 17–18.

Proof of the theorem that the locus of the points which divide the join of any two points on each of two null lines in Euclidean R_n in any constant ratio is a minimal X_2. This theorem goes back essentially to E. E. Levi [Ann. R. Scuola Norm. Pisa **10** (1908)]. *D. J. Struik* (Cambridge, Mass.)

2730:

Komissaruk, A. M. Fields of vectors in a two-dimensional Riemannian space. Minsk. Gos. Ped. Inst. A. M. Gor'k. Uč. Zap. **5** (1956), 15–40. (Russian)

Soit donnée une surface $V_2\subset E_3$, soient $\{r_1, r_2, n\}$ (n étant le vecteur normal) les repères locaux. L'A. introduit la notion de dérivée d'une fonction φ (vecteur a) selon le vecteur x par $\varphi'x=\varphi_{|\alpha}x^\alpha$ ($a'x=a^\alpha{}_{|\beta}x^\beta r_\alpha$), il démontre les identités de Ricci, Bianchi, Bianchi-Padova et les formules fondamentales pour le gradient, la divergence et le rotationnel. L'équation differentielle $a'x=\lambda x+\nu'x\cdot\tilde{a}$ pour les scalaires λ, ν et le vecteur a ($\tilde{a}=[na]$, x est un vecteur arbitraire) possède une solution pour chaque surface: on trouve φ, ν, t, μ, λ, a des équations

$$(*)\qquad \Delta_2\varphi+K=0,$$
$$\nabla\nu=(\nabla\varphi)'\tilde{\nabla}\varphi:(\nabla\varphi)^2-\tilde{\nabla}\varphi,\quad t=\nabla\varphi:\sqrt{(\nabla\varphi)^2},$$
$$\mu=e^\varphi,\quad \lambda=e^\varphi\sqrt{(\nabla\varphi)^2},\quad a=\mu t.$$

A chaque réseau isotherme de V_2 il correspond une solution de (*) (à une constante près) et inversement chaque φ satisfaisant à (*) détermine le réseau isotherme à une rotation de la direction du réseau d'un angle constant près [voir Weise, Math. Z. **46** (1940), 665–691; MR **2**, 161]. L'angle des directions des deux réseaux et la différence des fonctions φ correspondantes sont deux fonctions harmoniques conjuguées; ν est l'angle de la direction du réseau isotherme avec celle du réseau orthogonal contenant la couche $\varphi=$const; le réseau $(\nabla\varphi, \tilde{\nabla}\varphi)$ est isotherme si et seulement si ν est harmonique.

La condition nécessaire et suffisante pour que les courbes $\varphi=$const soient géodésiquement parallèles et à courbure géodésique constante est (**) $\Delta_1\varphi=f(\varphi)$, $\Delta_2\varphi=$

$F(\varphi)$. La fonction φ pour laquelle (*), (**) valent existe précisément pour les surfaces qui sont en déformation avec les surfaces de rotation. Enfin on trouve les solutions de (*) fonctionellement dépendantes de K.

A. Švec (Prague)

2731a:

Mineo, Massimo. Del variare della curvatura geodetica d'una curva nella rappresentazione d'una superficie su di un'altra. Matematiche, Catania **11** (1956), 111–116 (1957).

2731b:

Marussi, Antonio. Ancora sulla variazione della curvatura geodetica di una curva nella rappresentazione di una superficie su di un'altra. Matematiche, Catania **11** (1956), 163–167 (1957).

These authors have already written on this subject [Marussi, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16** (1954), 478–483; MR **16**, 745; Mineo, Matematiche, Catania **11** (1956), 1–7; MR **18**, 229]. Here, Mineo argues the relative merits of the two papers. Marussi replies. No new results or points of view are presented.

A. Schwartz (New York, N.Y.)

2732:

Komissaruk, A. M. Nets of equal paths and nets of equal curvatures. Minsk. Gos. Ped. Inst. A. M. Gor'k. Uč. Zap. **7** (1957), 11–20. (Russian)

This is a report based mainly on the work of J. Dubnov [Uč. Zap. Moskov Gos. Univ. **100** Mat. Tom. I (1946), 212–216; MR **12**, 282] and K. H. Weise [Math. Z. **46** (1940), 665–691; MR **2**, 161], as well as on the author's dissertation (1953). Nets of equal curvature are characterized by the property that, at every point of the surface, the geodesic curvatures of the two net curves are equal in absolute value.

D. J. Struik (Belmont, Mass.)

2733:

Saban, Giacomo. Estensione alle superficie rigate chiuse di un teorema di Fenchel ed Avakumovic. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **21** (1956), 245–251 (1957). (Turkish summary)

The theorem referred to in the title states that a closed spherical curve is the indicatrix of the tangents of a simple closed spherical curve if and only if it bisects the surface of the sphere [cf. W. Fenchel, Tôhoku Math. J. **39** (1934), 95–97; V. G. Avakumović, Srpska Akad. Nauka. Zb. Rad. **7**, Mat. Inst. **1** (1951), 101–108; MR **14**, 202]. Let R be a closed ruled surface, and denote by J the spherical indicatrix of its generators. Then the closed ruled surface R^* generated by the normals of R at the points of the line of striction of R has the following properties. (a) The spherical indicatrix of its generators coincides with the indicatrix of the tangents of J and hence bisects the surface of the sphere. (b) The orthogonal trajectories of its generators are closed. Conversely, it is shown that if a closed ruled surface R^* has these two properties, its generators are the normals along the line of striction of another closed ruled surface R. The proof is based on the representation of the lines of euclidean 3-space by dual vectors.

W. Fenchel (Copenhagen)

2734:

Kovancov, N. I. Des champs des vecteurs associés à un complexe reglé. Ukrain. Mat. Ž. **10** (1958), no. 1, 37–58. (Russian. French summary)

The author uses the ideas and the technique of some of his previous works [e.g. Dokl. Akad. Nauk SSSR **90** (1953), 125–128; Ukrain. Math. Ž. **5** (1953), 312–324; MR **15**, 348] to investigate all possible vector fields connected with a line complex such that the vectors at each point are parallel to one of the lines of the complex. The resulting vector fields depend on three parameters. One of them, the so-called central field, can be considered as the field of normal vectors of a non-holonomic surface σ, so that two quadratic forms can be introduced. When σ is holonomic, the complex degenerates into a one-parameter family of normal congruences. This decomposition depends on a function of two parameters. Special cases are those in which the normal surfaces are minimal surfaces, surfaces of constant curvature, or surfaces of Lamé. The paper finishes with fields associated with a complex of which the central lines are circles.

D. J. Struik (Belmont, Mass.)

2735:

Karapetyan, S. E. Linear complexes of developable surfaces of congruences. Akad. Nauk Armyan. SSR Dokl. **25** (1957), no. 3, 97–100. (Russian. Armenian summary)

To a ray of a congruence in ordinary projective space correspond two linear complexes A and A', each of which is determined by five consecutive generators of the developable surfaces of the congruence. When and only when the congruence belongs to a linear complex, the two complexes A and A' coincide with it.

D. J. Struik (Belmont, Mass.)

2736:

Fava, Franco. Nuove generalizzazioni geometriche dell'equazione di Jacobi. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 371–391.

This reviewer examined the differential equations obtained by giving an application $T:x\to\bar{x}$ of a projective plane into itself and by considering at each point x the differential element E_1 having x as its center and the join $x\bar{x}$ as its line: if T is a collineation, Jacobi's equation is obtained [Ann. Mat. Pura Appl. (4) **39** (1955), 15–24; MR **18**, 65].

The author provides various generalizations of this procedure.

Given T (on the plane) and a linear system ∞^{n+1} of algebraic curves, to a point x we may associate the differential elements E_n of order n, with center at x and belonging to the curves of the system through Tx: the set of E_n so determined satisfies a differential equation of order n.

To this type belong for $n=1, 2, 3$ the equations

$$y'=f(x, y);$$
$$y''=H_0y'^3+H_1y'^2+H_2y'+H_3 \quad (H_i=H_i(x, y));$$
$$y'''=F+Gy''+Hy''^2,$$

F, G, H particular rational functions of y', already considered by A. Terracini [Univ. Nac. Tucuman. Rev. A. **3** (1942), 195–234; **6** (1948), 273–287; MR **4**, 256; **10**, 539].

Similarly in a projective 3-space, given $n+1$ transformations T_i: $x\to\bar{x}_i$ and a linear system of ∞^N, $N=\frac{1}{2}(n+1)(n+2)$, algebraic surfaces, to each point x we may associate the superficial differential elements (or caps) of order $n(\sigma_2{}^n)$ having their centers at x of the surfaces of the given system containing x and $\bar{x}_i$. These elements are the integral elements of a (particular) system of $n+1$ partial differential equations of order n.

Particular systems (specializing n, T_i, and the system of surfaces) and their geometric properties are examined.

E. Bompiani (Rome)

2737a:

Anglès d'Auriac, Paul. Sur la notation de l'algèbre tensorielle. C. R. Acad. Sci. Paris **245** (1957), 1489–1491.

2737b:

Anglès d'Auriac, Paul. Sur la notation de l'algèbre tensorielle. C. R. Acad. Sci. Paris **245** (1957), 1584–1585.

2737c:

Angles d'Auriac, Paul. Sur la notation de l'analyse tensorielle. C. R. Acad. Sci. Paris **245** (1957), 1685–1687.

L'auteur expose un formalisme graphique pour le calcul tensoriel utilisant des schémas et des flèches.

G. Papy (Brussells)

2738:

Singh, H. D. An extension of a covariant differentiation process. Tensor (N.S.) **7** (1957), 137–140.

Consider a curve $x^i=x^i(t)$ in an n-dimensional manifold and put $x^{(r)i}=d^rx^i/dt^r$. Suppose that there is given a tensor $T^{\alpha\cdots}_{\beta\cdots}$ whose components are functions of x^i, $x^{(1)i}$, $\cdots$, $x^{(m)i}$. H. V. Craig [Bull. Amer. Math. Soc., **37** (1931), 731–734] introduced a covariant derivative of $T^{\alpha\cdots}_{\beta\cdots}$ of the form

$$T^{\alpha\cdots}_{\beta\cdots,\,(m-1)\gamma}-mT^{\alpha\cdots}_{\beta\cdots,(m)\lambda}\left\{{\lambda \atop \gamma}\right\},\quad m\geqq 2,$$

where the index $(r)\lambda$ following a comma denotes partial derivative with respect to $x^{(r)\lambda}$. M. M. Johnson [Bull. Amer. Math. Soc. **46** (1940), 269–271; MR **1**, 273] introduced a covariant derivative of the form

$$T^{\alpha\cdots}_{\beta\cdots,\,(m-2)\gamma}-(m-1)T^{\alpha\cdots}_{\beta\cdots,\,(m-1)\lambda}\left\{{\lambda \atop \gamma}\right\}-\frac{m(m-1)}{2}T^{\alpha\cdots}_{\beta\cdots,\,(m)\lambda}\left|{\lambda \atop \gamma}\right|$$

for $m=3$.

The present author gives a covariant derivative of the above form for $m=4$. *K. Yano* (Paris)

2739:

Lalan, V. Quelques applications géométriques de la différentiation extérieure. Bul. Inst. Politech. Iaşi **4** (1949), 108–122.

The paper consists of numerous applications of Cartan's exterior differentiation to the differential geometry of surfaces. Some of these applications are classic. Geometric results are obtained concerning the geodesic curvature of a curve, the total curvature of a surface, important parametric nets on a surface, surfaces of Weingarten, isothermal nets of curves, isothermic surfaces, surfaces of O. Bonnet.

A. Fialkow (Brooklyn, N.Y.)

2740:

Nožička, František. Sur le contact des variétés dans un espace affine linéaire. Časopis Pěst. Mat. **83** (1958), 171–201. (Czech. Russian and French summaries)

Pour que deux variétés $x=x(u_1, \cdots, u_p)$, $y=y(v_1, \cdots, v_p)$ d'espace affine E_n aient un contact géométrique d'ordre k (au moins) au point $\underset{0}{x}=x(\underset{0}{u})=y(\underset{0}{v})$ commun, il faut et il suffit: Soient $\underset{}{a}^s$ $(s=1, \cdots, n-p)$ des vecteurs constants dans E_n avec $[\underset{0}{x}_1, \cdots, \underset{0}{x}_p, a^1, \cdots, a^{n-p}]\neq 0\neq [\underset{0}{y}_1, \cdots, \underset{0}{y}_p, a^1, \cdots, a^{n-p}]$. Le système $y(v)-x(u)=\sum_{s=1}^{n-p}\underset{s}{\lambda}a^s$ définit une correspondance entre les variétés et les scalaires $\underset{s}{\lambda}$; on a $(d^i\underset{s}{\lambda})_0=0$ pour $i=1, \cdots, k$; $s=1, \cdots, n-p$. *A. Švec* (Prague)

2741:

Raifeartaigh, L. Fermi coordinates. Proc. Roy. Irish Acad. Sect. A. **59** (1958), 15–24.

Schouten and Struik showed that a necessary and sufficient condition for the existence of Fermi coordinates (coordinates in which the components of the symmetric connection of an N-dimensional affine space, V_n, vanish over an M-dimensional subspace U_m) is that N linearly independent vector fields exist which are parallel for all directions of V_n at the points of U^m [J. A. Schouten and D. J. Struik, Einführung in die neueren Methoden der Differentialgeometrie, I, Noordhoff, Groningen, 1935; p. 106]. In 1954 Schouten raised the question as to whether a weaker necessary and sufficient condition could be given, namely, that the N linearly independent vector fields need only be parallel for all directions of U_m at the points of U_m [J. A. Schouten, Ricci-calculus, 2d ed., Springer, Berlin-Göttingen-Heidelberg, 1954; p. 169]. The author of this article proves that the answer to Schouten's question is "yes". To establish this theorem, the author finds a necessary and sufficient condition for a subspace of a given space to be of distant parallelism and shows that along such a subspace and only such a subspace will a space admit a Fermi coordinate system. The construction of the Fermi coordinate system is given in two steps: (1) by use of a set of N parallel vectors a coordinate system is defined over the U_m, and (2) this coordinate system is extended into the neighboring part of V_n by use of geodesic coordinates. The transformation formulae for these coordinates and the inverse transformation are given explicitly. Finally, the author notes that the pseudo-energy tensor of relativity vanishes in such a coordinate system and considers the possibility of such coordinates in an Einstein universe.

N. Coburn (Ann Arbor, Mich.)

2742:

Dumitraş, Viorel. Sur les espaces A_3 à groupe maximum. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **6** (1957) no. 13, 27–43. (Romanian. French and Russian summaries)

The author determines the A_3 with transitive groups $\mathfrak{G}_9$ and $\mathfrak{G}_8$ of automorphisms. There are two categories of (non-euclidean) A_3 with a $\mathfrak{G}_9$ and there is one with a $\mathfrak{G}_8$, found by G. Vrănceanu. The A_n with $\Gamma_{h1}{}^h=(\varepsilon x^1+\alpha+)\beta/M$, $\Gamma_{1h}{}^h=(\varepsilon x^1+\alpha-\beta)/M$, $\Gamma_{11}{}^1=2(\varepsilon x^1+\alpha)/M$, $M=1+\varepsilon(x^3)^2$, $\varepsilon=0, \pm 1$, α and β constants, $h=2, \cdots, n$, have a $\mathfrak{G}$ with n^2 parameters. It is also shown that there are 2 classes of A_n with torsion and a group of n^2-2n+6 parameters; these considerations are extended to certain A_n with $(n-p)n-p+3+n(p-2)+6$ parameters, (for $p=n$ and $p=3$ this number becomes n^2-2n+6).

D. J. Struik (Belmont, Mass.)

2743:

Dumitraş, Viorel. Détermination des espaces A_3 à groupe $\mathfrak{G}_7$. Acad. R. P. Romîne. Stud. Cerc. Mat. **8** (1957), 183–234. (Romanian. Russian and French summaries)

Where in the paper quoted above the A_3 with transitive $\mathfrak{G}_9$ and $\mathfrak{G}_8$ have been determined, the present paper classifies those with transitive $\mathfrak{G}_7$. Among them are 3 classes which are symmetric and not projectively euclidean. These A_3 are sub-projective spaces of the first order. They can be generalized into A_n $(n\geqq 3)$ which are $(n-2)$-times projective with a group $\mathfrak{G}_k$, $k=n^2-2n+4$. The Russian and French summaries are quite satisfactory.

D. J. Struik (Belmont, Mass.)

2744:

Dumitraş, Viorel. Les groupes de mouvement à 6 paramètres des espaces A_3. Acad. R. P. Romîne. Stud. Cerc. Mat. **8** (1957), 303–342. (Romanian. Russian and French summaries)

This paper is dedicated to the A_3 with a transitive group $\mathfrak{G}_6$ of automorphisms. For their determination it is required to start with the stability group of the origin, which can be considered as a subgroup of the central affine group in three variables. There are only 10 non-similar such groups which can serve as total stability groups of an A_3 which is not Riemannian and has no group of motion with more than 6 parameters. This leads to ten classes of A_3, some symmetrical and some with torsion. One class can be generalized into a class of A_n ($n \geqq 3$) with a group of n^2-2n+3 parameters, another class into A_n ($n \geqq 3$) with semi-symmetric connection and a group of n^2-n parameters.

D. J. Struik (Belmont, Mass.)

2745:

Vranceanu, G. Sur les espaces à connexion affine localement euclidiens et les transformations crémoniennes entières. Rev. Math. Pures Appl. **2** (1957), 111–125.

Necessary and sufficient condition that a locally euclidean affine space A_n with its $\Gamma_{jk}{}^i$ entire functions of the coordinates $x^1, \cdots, x^n$ be globally equivalent to a euclidean space E_n is that the solutions of the system $d^2x^i/dt^2 = \Gamma_{jk}{}^i(dx^j/dt)(dx^k/dt)$ be entire functions in t, hence that the image in E_n of every selfparallel curve of A_n be a straight line in its entirety. This theorem is used to show that a locally euclidean A_n with constant $\Gamma_{jk}{}^i$ is then and only then equivalent to E_n if the finite equations of its selfparallel curves be polynomials in t. Then entire Cremona transformations $u^i = u^i(x^1, \cdots, x^n)$ are introduced as transformations where the u^i are polynomials as are the x^i, in u^i. ($\Delta = \mathrm{Det}|\partial u^i/\partial x^j|$ has to be constant [see B. Segre, 3d Congress Soviet Mathematicians, Moscow, 1956, ch. VI].) It is shown that to an entire Cremona transformation u^i, by means of the equations $\partial^2 u^i/\partial x^j \partial x^k = -\Gamma_{jk}{}^s(\partial u^i/\partial x^s)$, corresponds a locally euclidean A_n with polynomial connection, globally equivalent to E_n. Then A_n are studied corresponding to Jonquières transformations [H. W. E. Jung, J. Reine Angew. Math. **184** (1942), 161–174; MR **5**, 74] and to entire Cremona transformations which for $n=3$ can be written $u^1 = x^1$, $u^2 = ax^2 + bx^3 + c$, $u^3 = \alpha x^2 + \beta x^3 + \gamma$, where $a, b, c, \alpha, \beta, \gamma$ are polynomials in x^1 ($a\beta - \alpha b \neq 0$).

D. J. Struik (Belmont, Mass.)

2746:

Tevzadze, G. N. On the straight lines of a canonical pencil. Soobšč. Akad. Nauk Gruzin. SSR **18** (1957), 513–519. (Russian)

Following G. V. Bušmanova and A. P. Norden [Dokl., Akad. Nauk SSSR N.S. **60** (1948), 1309–1312; MR **10**, 478] and A. I. Čahtauri [ibid. **59** (1948), 1257–1259; MR **9**, 531], formulas are given for the geometry of a surface in three-dimensional projective space, normalized in the sense of Norden. An invariant expression for the vectors which determine the lines of a canonical pencil is obtained. A new interpretation of these lines is reached by means of the concept of the mean (middle) line of Green of a given net g_{ij} on the surface. This particular line of Green can then be related to the normalization of Lie, the projective normal of Fubini and the directrix of Wilczynski.

D. J. Struik (Belmont, Mass.)

2747:

Italiani, Mario. L'intorno di un punto unito a Jacobiano nullo in una trasformazione puntuale fra spazi. Boll. Un. Mat. Ital. (3) **12** (1957), 254–263.

Sia O un punto unito in una trasformazione puntuale T tra due spazi lineari sovrapposti S_r. Se in O il determinante jacobiano è nullo di caratteristica massima, la T induce nella stella di rette di centro O una proiettività singolare avente come spazi singolari una retta t ed un iperpiano α, e quindi se $t \notin \alpha$ una proiettività non singolare ω tra gli S_{r-2} per O ed appartenenti ad α. Nelle condizioni dette e nel caso che la ω sia generale l'Autore studia l'intorno del punto O determinando riferimenti intrinseci per la T. Per $r=3$ si studia anche il caso in cui la ω non sia generale ed il caso in cui la caratteristica del determinante jacobiano sia uguale ad 1.

C. Longo (Parma)

2748:

Kelly, Paul; and Straus, Ernst. Curvature in Hilbert geometries. Pacific J. Math. **8** (1958), 119–125.

Let C be a closed convex curve in E^2 which does not have two non-collinear segments as subarcs. Using C exactly as an ellipse in the Klein model of hyperbolic geometry to define a distance $h(p, q)$ in the interior D of C by means of cross-ratio, we obtain a Hilbert geometry. The geodesics are carried by the euclidean lines. If, for a fixed point p, any two points x_i, and the midpoints y_i of p and x_i, either always $h(x_1, x_2) \geqq 2h(y_1, y_2)$ or always $h(x_1, x_2) \leqq 2h(y_1, y_2)$, then p is a projective centre of C (i.e., with a suitable projectivity π, the point $p\pi$ is the affine centre of $c\pi$). Consequently, if the hypothesis is satisfied for every p in D, then the metric is hyperbolic.

H. Busemann (Cambridge, Mass.)

2749a:

Stavroulakis, Nicias. Nappeslogarithmiques d'un espace riemannien à deux dimensions. C. R. Acad. Sci. Paris **246** (1958), 1149–1152.

2749b:

Stavroulakis, Nicias. Les points logarithmiques et les points coniques dans les espaces de Riemann à deux dimensions. C. R. Acad. Sci. Paris **246** (1958), 1368–1371.

These two papers are concerned with properties of a surface in the neighbourhood of a special type of singularity analogous to the vertex of a half-cone. The Gauss-Bonnet theorem is extended to apply to a region of a surface which contains singularities of this type.

T. J. Willmore (Liverpool)

2750:

Fedenko, A. S. Limit spaces. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 235–240. (Russian)

If we consider a two-dimensional sphere with Riemannian metric $ds^2 = R^2(1+x^2+y^2)^{-2}(dx^2+dy^2)$, then by introducing coordinates $x' = Rx$ and $y' = Ry$ and letting R become infinite we obtain the metric $ds^2 = dx'^2 + dy'^2$ in the Euclidean plane. Essentially, this same limiting process can be applied to the group of motions of this sphere, which is a symmetric space, and the limit process gives the group of motions of the plane. Thus, we pass in the limit from a simple group to one which is not semi-simple. The author generalizes this limiting process to more general symmetric spaces, and uses it to obtain an example of an irreducible symmetric space whose group of motions is not semi-simple.

W. M. Boothby (Evanston, Ill.)

2751:

Kručkovič, G. I. On semireducible Riemannian spaces.

Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 862–865. (Russian)

A Riemannian V_n is semireducible if its ds^2 can be cast into the form $ds^2=ds_0^2+\sigma ds_1^2$, where $ds_0^2=g_{ij}dx^idx^j$, $i, j=1, 2, \cdots, q$, g_{ij} a function of the x^i; $ds_1^2=a_{\alpha\beta}dx^\alpha dx^\beta$, $\alpha, \beta=q+1, \cdots, n$, $a_{\alpha\beta}$ a function of the x^α and σ dependent only on the x^i. Such V_n have lately been investigated by H. Wakakuwa [Tôhoku Math. J. **6** (1954), 121–134; MR **16**, 956], H. L. De Vries [Math. Z. **60** (1954), 328–347; MR **16**, 168] and A. S. Solodovnikov [Uspehi Mat. Nauk (N.S.) **11** (1956), no. 4(70), 45–116; MR **18**, 930] in connection with certain transformation groups (conformal, geodesic, motion). One of the theorems stated in the present paper is that V_n are semireducible if and only if there exists a tensor $A_{ab}=A_{ba}$, not proportional to the metrical tensor, and a gradient vector u_a for which

(1) $A_{ab,c}=-\frac{1}{2}(u_aA_{bc}+u_bA_{ac})$; (2) $A_{ac}A_b{}^c=A_{ab}$

(the first part of this condition is sufficient for the case $ds^2>0$). This theorem generalizes a result of P. A. Širokov, [Izv. Kazan. Fiz.-Mat. Obšč. (3) **11** (1938), 3–7]. Some questions concerning the uniqueness of the representation of the metric are also taken up.

D. J. Struik (Cambridge, Mass.)

2752:

Kručkovič, G. I. On motions in semi-reducible Riemann spaces. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 6(78), 149–156. (Russian)

This is a report on recent work done by several authors on semi-reducible V_n, as well as an original exposition, primarily from the point of view of groups of motions. Apart from the papers quoted in the preceding review, we find references to K. Yano [Trans. Amer. Math. Soc. **74** (1953), 260–279; MR **14**, 688], I. P. Egorov [Dokl. Akad. Nauk SSSR **103** (1955), 9–12; MR **17**, 405] and others, as well as to work done by the author himself [ibid. **108** (1956), 583–586; MR **19**, 312].

D. J. Struik (Cambridge, Mass.)

2753:

Klingenberg, W. On the structure of compact Riemannian manifolds. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 586–588.

The author in this paper announces two main theorems concerning compact even dimensional Riemann manifolds with positive sectional curvature which is bounded away from zero. The first is essentially a tool theorem which gives detailed information concerning least lengths of closed geodesics and diameters of regions in which there exist unique shortest geodesics joining two points. These results are then applied to obtain some improvements on the theorem of Rauch [Ann. of Math. (2) **54** (1951), 38–55; MR **13**, 159]. The author obtains the following result. Let M be a complete Riemann manifold of even dimension. If the sectional curvature K of M satisfies the inequalities $0<hL<K\leq L$, where $h\sim 0.54$ is the solution of the equation $\sin \pi\sqrt{h}=h$, then M is diffeomorphic either with the sphere or with the elliptic space. In Rauch's original version of this theorem $h\sim 0.74$. It is known that for dim $(M)>2$, that h cannot be smaller than 0.25.

L. Auslander (Bloomington, Ind.)

2754:

Berger, Marcel. Sur certaines variétés riemanniennes à courbure positive. C. R. Acad. Sci. Paris **247** (1958), 1165–1168.

This paper, as that of Klingenberg, reviewed above, continues the study of even-dimensional compact spaces, such that the sectional curvature K is positive and bounded away from zero. In this work the author assumes that $0<\frac{1}{4}k<K\leq k$ and then shows that the space is homologically a sphere. Further, if $0<\frac{1}{4}k\leq K\leq k$, then either the space is an homology sphere or it is isometric with a symmetric Riemann space of rank 1. The proofs are all essentially based on Morse's theory of the calculus of variations in the large.

L. Auslander (Bloomington, Ind.)

2755a:

Širokov, P. A. On a certain type of symmetric spaces. Mat. Sb. N. S. **41**(83) (1957), 361–372. (Russian)

2755b:

Şirokov, P. A. On a certain type of symmetric spaces. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **12** (1958), no. 1 (24), 5–18. (Romanian)

[Russian and Romanian versions of the same paper].

The paper contains material on the geometry of a certain class of symmetric spaces, which was found in the papers of P. A. Širokov, who died in 1944, and was prepared for publication with an added page of comments by A. P. Širokov, A. Z. Petrov and B. A. Rozenfel'd. Generally speaking, a symmetric Riemannian space, according to one definition, is a Riemannian manifold whose curvature tensor, $R_{\alpha\beta\gamma\delta}$, has covariant derivative identically zero. The symmetric spaces considered in this paper, in addition to being of even dimension, have the added property that at some point we have satisfied:

$$\tfrac{1}{3}R_{\alpha\beta\gamma\delta}=\alpha(a_{\alpha[\gamma}a_{\delta]\beta}+b_{\alpha[\gamma}b_{\beta\delta]}+b_{\alpha\beta}b_{\gamma\delta}),$$

where $a_{\alpha\beta}$ is the metric tensor at the point and $b_{\alpha\beta}$ is a skew symmetric tensor related to $a_{\alpha\beta}$ by $b_\alpha{}^\sigma b_{\beta\sigma}=a_{\alpha\beta}$. The author uses this relation to develop $g_{\alpha\beta}$, the metric tensor, at the point in a power series which he is actually able to sum, i.e., to write in suitable coordinates z^α in the form

$$g_{\alpha\beta}=a_{\alpha\beta}+\phi z_\alpha z_\beta-k^{-2}\eta_\alpha\eta_\beta,$$

where $\phi=(k^2-\sigma)^{-1}$, $\sigma=a_{\alpha\beta}z^\alpha z^\beta$, $z_\alpha=a_{\alpha\sigma}z^\sigma$, $\eta_\alpha=b_{\alpha\sigma}z^\sigma$, and $\alpha=k^{-2}$. After further minor changes of coordinates to alter this expression somewhat, he finds expressions for the affine connection and curvature. Then, using these, he studies the geodesics, for which he gets a simple equation, and the trigonometry of these spaces, i.e., the relations between the angles and lengths of sides of a triangle. In the paper of B. A. Rozenfel'd, which appeared with this one and is reviewed below, the same results are obtained by different methods and extended to symmetric spaces of rank one, which include those defined above.

W. M. Boothby (Evanston, Ill.)

2756a:

Rozenfel'd, B. A. On the theory of symmetric spaces of rank one. Mat. Sb. N. S. **41**(83) (1957), 373–380. (Russian)

2756b:

Rozenfeld, B. A. On the theory of symmetric space of rank one. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **12** (1958), no. 1 (24), 19–27. (Romanian)

[Russian and Romanian versions of the same paper.]

The symmetric spaces of rank one having a simple group of motions are coset spaces of Lie groups and consist of five types: $A_n/A_{n-1}\times D_1$, B_n/D_n, $C_n/D_{n-1}\times A_1$, D_n/B_{n-1}, and F_4/B_4, which are, respectively, complex projective space $K_n(i)$, even dimensional sphere S_{2n}, quaternion projective space $K_{n-1}(i, j)$, odd dimensional sphere S_{2n-1}, and $K_2(i, j, l)$ the projective space of the

Cayley octaves. The spaces $K_n(i)$ are essentially those considered by P. A. Širokov in the paper reviewed above. In the present case the author extends and clarifies the results of Širokov, giving a unified account of the geometry (i.e. geodesics, lines, sectional curvature, and trigonometry) of all five types of spaces simultaneously. The proofs use, among other devices, the theory of Lie algebras and a fundamental theorem of E. Cartan on symmetric spaces.

Consider only the case $K_n(i)$, complex projective space, as a typical example. The metric used is the standard one of Fubini-Study. Submanifolds isometric to $K_1(i)$ are called lines; they are in fact isometric to the sphere S_2 in euclidean three dimensional space. Every geodesic is contained in such a line and is thus a great circle on a sphere. Each pair of tangent vectors determines a plane element which in general does not lie on a line; its deviation from this position is measured by an angle α defined by the author (for all the spaces mentioned). This angle of inclination of a plane element plays an important role in the formulas obtained. It vanishes when the plane element lies on a line. In particular, the sectional curvature K of a plane element is shown to be $K=r^{-2}(1+3\cos^2\alpha)$, where r is a fixed constant depending on the Riemannian metric chosen for the space. Formulas of trigonometry, extending those of P. A. Širokov above, and reducing in special cases to standard formulas such as the laws of sines and cosines of spherical trigonometry are obtained. For example, if a, b, c are the sides of a geodesic triangle, A the angle opposite a, and α the inclination (defined above) of the plane of the triangle, then we have

$$\cos(2a/r)=\cos(2b/r)\cos(2c/r)+\sin(2b/r)\sin(2c/r)\cos A-2\sin^2\alpha\sin^2(b/r)\sin^2(c/r)\sin^2 A;$$

when $\alpha=0$, the last term vanishes and this becomes the law of cosines of spherical trigonometry for a sphere of radius $r/2$. As mentioned, each result holds, word for word, for each of the spaces $K_n(i)$, $K_n(i,j)$ and $K_2(i, j, l)$. These results generalize results of J. L. Coolidge [Ann. of Math. **22** (1921), 11–28] and W. Blaschke and H. Terheggen [Rend. Sem. Mat. Univ. Roma (4) **3** (1939), 153–161; MR **1**, 261]. *W. M. Boothby* (Evanston, Ill.)

2757:

Allamigeon, André-Claude. Espaces homogènes symétriques harmoniques. C. R. Acad. Sci. Paris **246** (1958), 1004–1005.

If a symmetric space G/H is harmonic relative to a metric invariant under G then it is simply harmonic or G is semi-simple. Let m be the (-1)-eigenspace of the involutive automorphism σ (in the definition of G/H as a symmetric space). Then G/H is simply harmonic relative to a metric invariant under G if and only if m is in the maximal nilpotent ideal of G.

W. Ambrose (Cambridge, Mass.)

2758:

Mutō, Yosio. On some properties of affinely connected manifolds admitting groups of affine motions of order $r>n^2-pn$. Tensor (N.S.) 7 (1957), 86–96.

L'A. ottiene il seguente teorema: Condizione necessaria affinchè una varietà A_n a connessione affine, $n\geq 4$, senza torsione, ammetta un gruppo di movimenti affini di ordine $r>n^2-pn$, ove $p<(n-2)/2$, è che il suo tensore di curvature sia della forma:

$$R^{\lambda}{}_{\cdot\mu\nu\omega}=\delta_{\mu}{}^{\lambda}(P_{\nu\omega}-P_{\omega\nu})+\delta_{\nu}{}^{\lambda}P_{\mu\omega}-\delta_{\omega}{}^{\lambda}P_{\mu\nu}+A_{(1)}{}^{\lambda}B^{(1)}{}_{\mu\nu\omega}+\cdots+A_{(p)}{}^{\lambda}B^{(p)}{}_{\mu\nu\omega},$$

ove alcuni dei vettori e tensori al secondo membro possono essere nulli. Il risultato viene ottenuto per via algebrica. — considerando cioè le trasformazioni infinitesime del gruppo cercato in un punto fissato — servendosi del rango di opportune matrici formate colle componenti del tensore di curvatura, che appaiono nelle condizioni di integrabilità delle equazioni che dànno le trasformazioni infinitesime stesse. *V. Dalla Volta* (Rome)

2759:

Tashiro, Yoshihiro. On universal tensorial forms on a principal fibre bundle. J. Math. Soc. Japan **8** (1956), 247–255.

The notion of an affine connexion on a manifold M can be formulated in the following three ways, among others: (1) a "horizontal" subspace in the bundle B of bases over M, at each point of B; (2) a certain kind of 1-form on B; (3) a family of 1-forms on M, of a certain kind, each defined on an open in M, and related by certain transformations on the overlap of any two of these opens. This paper shows the equivalence of these three definitions for general connexions. It also considers certain formulas for covariant derivatives. *W. Ambrose* (Cambridge, Mass.)

2760:

Kobayaschi, Shoshichi. Theory of connections. Ann. Mat. Pura Appl. (4) **43** (1957), 119–194.

The aim of the author of this exposition on connection theory was to give an exposition of E. Cartan's theory in the terminology of fibre bundles, and of connections in the sense of Ehresmann. The methods are definitely modern in that coordinates and the accompanying index notation are used only to deduce some initial basic lemmas, while otherwise more intrinsic methods prevail. The theory is built up from the concept of manifold, the only further prerequisite being some knowledge of classical differential geometry for comparison purposes. The basic tools are of surprising simplicity, and are very few in number. They are quite well known to any contemporary differential geometer, but some (including the reviewer) may find that the treatment here is far simpler than was thought possible. The main tools are (1) taking the tangent bundle to a manifold, (2) extending a (differentiable) map of one manifold into another to the tangent bundles. — Especially in the theory of Lie groups acting on manifolds a great simplicity is obtained.

Following is a table of contents. Chapter I: Differentiable manifolds, Lie groups, vector fields, tangent bundle. Chapter II: Fibre bundles, soldering ("soudure"), homogeneous spaces. Chapter III: Differential forms. Chapter IV: Infinitesimal connections, holonomy groups, tangent connections, curvature, Lie algebra of a holonomy group, local holonomy groups. Chapter V: Cartan connections, completeness of Cartan connections, development procedure, weakly reductive fibre bundles, invariant Cartan connections, the existence of connections and the second countability axiom. Chapter VI: Transformation groups. The results imply that the mappings preserving a Cartan connection form a Lie group. If the connection is complete, its Lie algebra is just the Lie algebra of vector fields leaving invariant the connection; otherwise the former is a subalgebra of the latter. *A. Nijenhuis* (Seattle, Wash.)

2761:

Takasu, Tsurusaburo. Non-connection methods for the theory of principal fibre bundles as almost Kleinean geometries. Proc. Japan Acad. **33** (1957), 515–520.

The paper under consideration is best described as a review and unification of methods developed by the author in some previous papers [Yokohama Math. J. 2 (1954), 81–94; 4 (1956), 1–46, 119–146; MR 16, 1053; 19, 61; 20#739]. According to the author, "it will be shown firstly that the present author's theory of the respective principal fibre bundles (based on the II-geodesic curves) is substantially nothing other than the respective theory in the large of S. S. Chern or that of C. Ehresmann," and that it "provides us non-connection methods for the differentiable manifolds admitting infinitely many connections and that the results reduce to such an extent that the geometries under consideration become the corresponding almost Kleinian geometries."

A. Nijenhuis (Seattle, Wash.)

2762:

Kawaguchi, Akitsugu; and Katsurada, Yoshie. On a connection in an areal space. Bul. Inst. Politech. Iaşi **4** (1949), 369–385.

The authors establish a theory of connections in an areal space by a method which differs significantly from earlier treatments of this subject [see, e.g., E. T. Davies. J. London Math. Soc. **20** (1945), 163–170; MR **8**, 96]. The area of an m-dimensional subspace of an n-dimensional space X_n, given by the parametric equations $x^i = x^i(u^a)$ $(i=1, \cdots, n;\ a=1, \cdots, m)$, is defined by an m-tuple integral $\int \cdots \int f(x^i, \partial x^i/\partial u^a) d(u^1, \cdots, u^m)$, the integrand f being a given function having suitable transformation properties. This gives rise to the so-called areal space $A_n{}^m$. The authors confine themselves to the case $n=2$. By introducing a certain intrinsic tensor Λ_{ijkl} and subjecting it to certain conditions, they succeed in deriving suitable connection coefficients. This gives rise to a theory of torsion and curvature. The curvature tensors may be interpreted in a manner which is similar to the geometrical interpretation of the three well-known curvature tensors of Cartan in a Finsler space.

H. Rund (Durban)

2763:

Libermann, Paulette. Pseudogroupes infinitésimaux. Faisceaux d'algèbres de Lie associés. I. Généralités. C. R. Acad. Sci. Paris **246** (1958), 41–43.

An infinitesimal pseudogroup (i.p.) is a collection E of vectorfields, defined on open subsets of a manifold V_n, closed under linear combination and bracket, and local: a vectorfield defined on $\bigcup U_i$ belongs to E exactly if its restrictions to the U_i do. The germs of E define a sheaf $J^\lambda(E)$ of Lie algebras. Associated concepts are developed, e.g., the pseudogroup (p.g.) generated by E. Th. 1: If the p.g. $\bar{\Gamma}$, on V_{m+n}, prolongs the p.g. Γ, on V_n, with respect to p: $V_{m+n} \to V_n$, then there is an induced map of the i.p.'s, and a homomorphism of $J^\lambda(\bar{E})$ onto $p^{-1}(J^\lambda(E))$ with a certain kernel. Generalization to foliated manifolds, etc.

H. Samelson (Ann Arbor, Mich.)

2764:

Libermann, Paulette. Pseudogroupes infinitésimaux. Pseudogroupes infinitésimaux de Lie. C. R. Acad. Sci. Paris **246** (1958), 531–534.

Let $J^s(E)$ be the set of s-jets of the elements of E (as sections of the tangent bundle of V_n). E is a Lie i.p. if (a) it is complete of order q (i.e., q is the first integer such that E consists of all solutions of $J^q(E)$), (b) the $J^s(E)$, $0 \leq s \leq q$ are analytic varieties in an appropriate sense. Concepts of transitivity (the values of the fields in E at x span the tangent space), finite type, degree r ($J^r(E)$ isomorphic to $J^{r-1}(E)$). If Γ is a Lie p.g., then the associated i.p. is of the same order, type, degree and transitivity behavior. Theorems connecting transitive Lie i.p.'s and p.g.'s to global objects in the simply connected case.

H. Samelson (Ann Arbor, Mich.)

2765:

Libermann, Paulette. Pseudo groupes infinitésimaux. Applications aux G-structures. C. R. Acad. Sci. Paris **246** (1958), 1365–1368.

The concept of finite type and degree are extended to non-Lie i.p.'s. The global transformations contained in a p.g. Γ of finite type form a Lie group of dimension $\leq \dim \Gamma$ (Th. 1). The differential equations for the Killing fields of an arbitrary G-structure on V_n are developed. A necessary and sufficient condition for G is given under which all affine connections of a G-structure with the same torsion are identical. In this case the pseudogroup of local automorphisms of any G-structure is of finite type.

H. Samelson (Ann Arbor, Mich.)

2766:

Lelong-Ferrand, Jacqueline. Sur les champs de vecteurs définissant un groupe d'homéomorphismes d'une variété différentiable. C. R. Acad. Sci. Paris **245** (1957), 1491–1493.

The author in this note announces theorems which relate vector fields on manifolds to existence of global one parameter groups of homeomorphisms. The results are for a very weak class of regularity assumptions on the vector field and the manifold and for a very large class of manifolds. The necessary and sufficient conditions are, however, too technical to reproduce explicitly in a review.

L. Auslander (Bloomington, Ind.)

2767:

Wu, Guang-lei. On n-manifolds in Euclidean $2n$-space. Sci. Record **1** (1957), no. 1, 35–36.

Let $f: M \to E$ be a regular immersion of a closed oriented n-manifold M in Euclidean $2n$-space. Then M has a self-intersection number d_f defined by H. Whitney [Ann. of Math. (2) **45** (1944), 220–246; MR **5**, 273] and, moreover, the immersion determines an n-dimensional cohomology class, known as the normal characteristic class. The author gives an integral formula for d_f, and finds an exterior differential form which equals the normal characteristic cocycle in the sense of de Rham. No proofs are given.

W. M. Boothby (Evanston, Ill.)

2768:

Kobayashi, Shoshichi. Compact homogeneous hypersurfaces. Trans. Amer. Math. Soc. **88** (1958), 137–143.

In this paper the following theorem is proved: Let M be an n-dimensional compact homogeneous Riemannian manifold and assume that M can be imbedded isometrically as a hypersurface in $(n+1)$-dimensional Euclidean space; then M is isometric to a sphere. By homogeneous is meant that the group of isometries of M (which is necessarily a Lie group) acts transitively on M. In proving this result the author notes that since M is compact, it has a neighborhood U which is locally convex (supposing M to be actually imbedded in Euclidean space). From this it follows that U is rigid if $n \geq 3$; the case $n=2$ is easier and is treated separately. From this rigidity and the homogeneity it follows that the compact group of all isometries of M forms a subgroup of the motions of the Euclidean space. Being a compact subgroup it has a fixed point, and (for $n \geq 3$) the theorem follows easily. The paper includes a very neat version of the rigidity theorem for hypersurfaces in Euclidean space mentioned above.

W. M. Boothby (Evanston, Ill.)

2769:

Chern, Shiing-shen. A proof of the uniqueness of Minkowski's problem for convex surfaces. Amer. J. Math. **79** (1957), 949–950.

By modifying the proof of Herglotz for Cohn-Vossen's theorem on isometries of closed convex surfaces by deriving some integral formulas, the author proves the uniqueness of Minkowski's problem which states that a closed convex surface of class C'' in ordinary Euclidean space is determined up to a translation when the Gaussian curvature is strictly positive and given as a function of the unit normal vector. *C. C. Hsiung* (Bethlehem, Pa.)

2770:

Hsiung, Chuan-Chih. A uniqueness theorem for Minkowski's problem for convex surfaces with boundary. Illinois J. Math. **2** (1958), 71–75.

A convex surface in E^3 is the closure of a bounded relatively open connected set on the boundary of a convex region in E^3 bounded by a finite number of disjoint closed Jordan curves. For two convex surfaces S, S^* of class C^2 with positive Gauss curvature, let a differentiable homeomorphism H of S on S^* exist such that S and S^* have, at corresponding interior points, equal Gauss curvatures and parallel exterior normals. If H, restricted to the boundary of S, coincides with a translation of E^3, then it does so on all of S. The proof follows very closely Chern's proof for the uniqueness in Minkowski's problem, i.e. the same problem for closed S, S^*. — {In the author's historical remarks he mixes results on uniqueness with results on the existence of smooth solutions for smooth data.} *H. Busemann* (Cambridge, Mass.)

2771:

Rembs, Eduard. Infinitesimale Verbiegungen von Flächen in sich. Math. Nachr. **16** (1957), 134–136.

Let $\mathfrak{x}=\mathfrak{x}(u, v) \in C^{(2)}$, $\mathfrak{z}=\mathfrak{z}(u, v) \in C^{(1)}$. If (1) $\mathfrak{x}\to\mathfrak{x}+t\mathfrak{z}$ is an infinitesimal isometry of a surface S into itself, S is isometric to a surface of revolution [cf. Bianchi-Lukat, Differentialgeometrie, 2nd ed., Teubner, Leipzig-Berlin, 1910, p. 327]. In a suitable coordinate system, S then has the line element (2) $ds^2=du^2+G(u)dv^2$ and $\mathfrak{z}$ satisfies (3) $\mathfrak{z}\mathfrak{x}_u=(\mathfrak{z}\mathfrak{x}_u\mathfrak{x}_v)=0$. Suppose conversely S is a simply connected surface of positive curvature with the line element (2) such that the net of parameter lines is regular everywhere on S. Then there exists one and up to a constant factor only one $\mathfrak{z}$ such that (1) is an infinitesimal isometry of S into itself which satisfies (3) along the boundary of S. (3) will then hold everywhere on S. *P. Scherk* (Saskatoon, Sask.)

2772:

Busemann, Herbert. Similarities and differentiability. Tôhoku Math. J. (2) **9** (1957), 56–67.

Let xy stand for the distance between the points x and y in a G-space [Busemann, The geometry of geodesics, Academic Press, New York, 1955; MR **17**, 779]. A map $x\to x'$ is called a local similarity if it is interior and $\lim(x'y'/xy)=k$ ($x\to p$, $y\to p$, $x\neq y$) exists for each p, with $0<k<\infty$. The constant k is called the dilation factor. It is shown that a simply-connected G-space with a proper ($k\neq 1$) local similarity must be straight (geodesics are unique in the large). However, two two-dimensional examples (one in which Desargues' Theorem holds) are described to show that there are G-spaces not locally Minkowskian which admit similarities. A definition of differentiability of the metric is introduced which enables one to construct tangent spaces; but even the regularity condition R_p: prolongation (of geodesics) is unique in the normal tangential space T_p, fails to make such a T_p Minkowskian.

However, requiring continuous differentiability changes matters completely. Let a and b be points in a G-space, β a real number $\geqq 0$. Denote by $a(\beta, b)$ a point c such that $bc=\beta(ba)$ and $bc+ca=ba$ or $ba+ac=bc$. (In G-spaces such points always exist locally.) Then the space is called continuously differentiable at p if

$$\lim_{\nu\to\infty} \frac{a_\nu(\beta_\nu, p_\nu)b_\nu(\beta_\nu, p_\nu)}{\beta_\nu(a_\nu b_\nu)}=1$$

for $a_\nu\to p$, $b_\nu\to p$, $a_\nu\neq b_\nu$, $p_\nu\to p$ and $0<\beta_\nu<M$. Theorem: Let the G-space R possess a proper local similarity, and assume R to be continuously differentiable at a fixed point of the similarity. When $k<1$, R is Minkowskian: when $k>1$, then the universal covering space of R is Minkowskian.

In his book "Metric methods in Finsler spaces" [Princeton, 1942; MR **4**, 109] the author gave intrinsic differentiability conditions for G-spaces to be Finsler spaces. The present conditions, however, are simpler and enable him to prove the theorem: If a G-space is continuously differentiable and regular at p, then the normal tangential metric at p is Minkowskian. A corollary is that such a space is a topological manifold. This gives a new intrinsic metric characterization of symmetric Finsler spaces. The paper concludes with a discussion of the "recoverability" of the class of differentiability of the metric. *L. W. Green* (Minneapolis, Minn.)

2773:

Nasu, Yasuo. On asymptotes in a metric space with non-positive curvature. Tôhoku Math. J. (2) **9** (1957), 68–95.

Various unexpected phenomena occur in the theory of parallels or of rays and their co-rays in general geometries [see the reviewer's "The geometry of geodesics", Academic Press, New York, 1955; MR **17**, 779; in particular sections 22, 23, 40]. It is shown here, partly under the assumption that small circles are differentiable, that the behaviour is normal on surfaces which are, topologically, spheres punctured at $k\geqq 2$ points and are metrized as G-spaces with non-positive curvature (l.c., section 36). For example: Every ray S' containing a co-ray S to a ray R is a co-ray to R. If the union of all such S' (for fixed S) is not a straight line, it is a ray with an initial point p, called asymptotically conjugate to R. There are at least two different co-rays from p to R and at most k. The relation between ray and co-ray is symmetric. The totality $G(R)$ of all asymptotically conjugate points to R is investigated. For $k\geqq 3$ it is not vacuous and consists of a finite number of unbounded curves with a finite number of branch points. Examples are given to show that the assumption of non-positive curvature is essential for these results. Some differentiability properties of limit circles with R as central ray at points not in $C(R)$ are derived for smooth metrics. *H. Busemann* (Cambridge, Mass.)

2774:

Moór, Arthur. On the extremals of the generalized metric spaces. Bul. Inst. Politehn. Iaşi (N.S.) **2** (1956), no. 3–4, 19–26. (Russian and Romanian summaries)

Associating a covariant [contravariant] vector density (u) of weight $-p$ $[+p]$ with each point (x) of an n-manifold, we obtain a manifold L_n $[L_n^*]$ with vector densities as fundamental elements. Let $L(x, u)$ be a scalar function homogeneous in (u) of degree one defined in L_n $[L_n^*]$, then $L(x, u)$ determines the metric character

of L_n [L_n^*], which represents a general space introduced by J. A. Schouten and J. Haantjes [Monatsh. Math. Phys. **43** (1936), 161–176]. In this general space the fundamental metric tensor $g_{ik}(x, u)$ can be constructed of the second derivatives of $L^2(x, u)$. On making use of the notations by R. S. Clark [Proc. London Math. Soc. (2) **53** (1951), 294–309; MR **13**, 74] and E. T. Davies [ibid. **49** (1947), 241–259; MR **8**, 491], the author considers the elements of support (u) as functions of a set of independent variables (x, $\dot{x}$) which defines the tangential line element of a curve $x^i=x^i(t)$. Then the variational problem of $\delta\int (g_{ik}(x, u(x, \dot{x}))\dot{x}^i\dot{x}^k)^{\frac{1}{2}}dt=0$ is reduced to $\delta\int F(x, \dot{x})dt=0$, of which the extremals can be found in the usual way, i.e., from the differential equations of Euler-Lagrange. Thus he obtains a general form of the extremals in L_n [L_n^*] and shows that the results give some of the results of J. G. Freeman [Quart. J. Math. Oxford Ser. **15** (1944), 70–83; MR **6**, 188] as special cases.

A. Kawaguchi (Sapporo)

PROBABILITY

See also 2413, 2631, 2817, 2818, 2826, 2827, 2983, 3036.

2775:

Kudō, Hirokichi. On sufficiency and completeness of statistics. Sûgaku **8** (1956/57), 129–138. (Japanese)

Let $X=\{x\}$ be the sample space and $\Omega^X=\{P^X\}$ be a class of probability measures over X which is assumed to be dominated. Denote the decision space by $L=\{l\}$ and the loss associated with l and P^X by $l\cdot P^X\in R_+$. Let $y=T(x)$ be a statistic and Φ^T be the class of decision functions which are T-measurable.

Consider a finite subset $\omega=\{P^X{}_1,\cdots,P^X{}_n\}$ of Ω^X and the convex set $L=\{l\cdot P^X{}_i|l\in L,\ i=1, 2, \cdots, n\}$, which is contained in the positive quadrant $R_+{}^n$ of n-space R^n. If there exists a decomposition $R^n=R_1\times R_2$; $R_1\perp R_2$, for which there is a supporting function $\mathrm{H}(\xi, L_\omega)$ such that $H(\xi, L_\omega)=H(\xi^1, L_\omega)+H(\xi^2, L_\omega)$, where $\xi^1\in R_1$, $\xi^2\in R_2$ and $\xi=\xi^1+\xi^2$, then L_ω is said to be reducible in R_1 and R_2. When L_ω is not reducible in any pair of subspaces of R^n, it is said to be irreducible. The paper consists of four paragraphs of which the first two give an excellent review of the previous works and the third gives lemmas on convex set in finite euclidean space. Generalizing the results of Elfving [Ann. Accad. Sci. Fenn. Ser. A. I. Math.-Phys. no. 135 (1952); MR **14**, 998], Bahadur [Ann. Math. Statist. **26** (1955), 286–293; MR **16**, 1133] and the author [Nat. Sci. Rep. Ochanomizu Univ. **4** (1954), 151–163; MR **16**, 730], he states the following main theorem in § 4: Assume that for any pair $P^X{}_1$, $P^X{}_2$ of measures in Ω^X we can add a finite set $P^X{}_3, \cdots, P^X{}_n$, such that L_{ω_0} (where $\omega_0=\{P^X{}_1, P^X{}_2, P^X{}_3, \cdots, P^X{}_n\}$) has a supporting point in any direction, and is irreducible. Then the essential completeness of Φ^T implies the sufficiency of $T(x)$. It seems to be necessary, besides the above mentioned conditions, to add the following: the carrier of the density of P^X is identical for all P^X in Ω^X except for a null set for the dominating measure of Ω^X.

T. Kitagawa (Fukuoka)

2776:

Lukacs, Eugene. Les fonctions caractéristiques analytiques. Ann. Inst. H. Poincaré **15** (1957), 217–251.

Let $F(x)$ be a probability distribution function (p.d.f.); the corresponding characteristic function is defined for all real t by $f(t)=\int_{-\infty}^{+\infty}\exp(itx)dF(x)$. A characteristic function $f(t)$ is called analytic if there is a function $A(z)$, analytic in a circle $|z|<\rho$, such that for some $\Delta>0$ we have $A(t)=f(t)$ for all real t, $-\Delta<t<\Delta$. The present paper is a systematic study of properties of analytic characteristic functions. It contains a wealth of material, including the derivation of many known theorems, as well as a number of important new results. Among the latter one may mention: a simple necessary and sufficient condition for a p.d.f. $F(x)$ to have an analytic characteristic function (Theorem 3.1); a necessary and sufficient condition for a random variable with an analytic characteristic function to have a range which is bounded from the left or from the right (Theorem 3.2); a theorem stating that, under certain very general assumptions, if an analytic characteristic function can be factored, then each factor is again an analytic characteristic function (Theorem 4.1). By presenting these new and old results in a well-planned sequence, the author attains a remarkable economy and lucidity of presentation of this important and, previously, rather difficult area of probability theory.

Z. W. Birnbaum (Seattle, Wash.)

2777:

Driml, Miloslav; et Hanš, Otto. Sur les positions typiques dans un espace distancié. C. R. Acad. Sci. Paris **246** (1958), 1653–1655.

Let X be a random variable with values in a metric space D having distance function ρ. Doss [Bull. Sci. Math. (2) **73** (1949), 48–72; MR **11**, 190] defined the class of means of X as the set $\Gamma(D, X)$ of elements γ for which $\rho(\gamma, \lambda)\leqq\mathfrak{M}[\rho(X, \lambda)]$ for all λ in D. The author gives an example in which D_0 is a subset of D, the value of X lies in D_0 with probability 1, $\Gamma(D, X)\subset D_0$, and yet $\Gamma(D, X)$ is a proper subset of $\Gamma(D_0, X)$.

Fréchet [Ann. Inst. H. Poincaré **10** (1948), 215–310; MR **10**, 311] defined an element γ_α of D as a typical element of X of order α if it lies in the set $\Gamma_\alpha(D, X)$ for which

$$\mathfrak{M}[\rho^\alpha(X, \gamma_\alpha)]=\inf_{\lambda\in D}\mathfrak{M}[\rho^\alpha(X, \lambda)].$$

Examples are given in which D is a Banach space and in which Γ_α differs from the Bochner integral of X.

J. L. Doob (Urbana, Ill.)

2778:

Teicher, Henry. Sur les puissances de fonctions caractéristiques. C. R. Acad. Sci. Paris **246** (1958), 694–696.

The following two theorems are proved. Th. 1 (corrected version): Let $\varphi(t)=(pe^{it}+1-p)^K$, K integral, $p>0$, and $\varphi(t)=\varphi_1{}^{\alpha_1}(t)\varphi_2{}^{\alpha_2}(t)$, $\alpha_i>0$, $i=1, 2$, where the $\varphi_i(t)$ are non-degenerate characteristic functions. Then $\varphi_i(t)=(pe^{it}+1-p)^{K_i}e^{itc_i}$, $c_1\alpha_1+c_2\alpha_2=0$, $K_1\alpha_1+K_2\alpha_2=K$. (The factor e^{itc_i} is missing in the paper.) Th. 2: Let $\varphi(t)$ be the characteristic function of a nondegenerate distribution whose mass is concentrated in the points $a+jh$, a real, $h\neq0$, $m\leqq j\leqq n$, j, m, n integral. A necessary condition for $\varphi^\alpha(t)$, $\alpha>0$, to be a characteristic function is that $(n-m)\alpha$ be integral.

G. E. Noether (Boston, Mass.)

2779:

Kemperman, J. H. B. Some exact formulae for the Kolmogorov-Smirnov distributions. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 535–540.

Let $x_1, x_2, \cdots, x_n; y_1, \cdots, y_{kn}$ be $(k+1)n$ independent random variables each having the continuous distribution

$F(x)$. Further, let

$$P_{n,kn}(x, y)=P(-x<F_n(s)-G_{kn}(s)<y \text{ for all } s),$$
$$Q_n(x, y)=P(-x<F_n(s)-F(s)<y \text{ for all } s),$$

where F_n and G_{kn} are the empiric distributions based on the first n and second kn random variables respectively. Exact values for $P_{n,kn}$ and Q_n are obtained and from these the corresponding one-sided distributions corresponding to $x=\infty$ or $y=\infty$ are computed. Although other expressions for these probabilities have been obtained the present results which are expressed in terms of the generating functions for $P_{n,kn}$ and Q_n would seem to be more useful in most cases.

J. Blackman (Syracuse, N.Y.)

2780:

Statulyavičus, V. A. A local limit theorem for nonhomogeneous Markoff chains with an enumerable number of possible states. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 872–873. (Russian)

An asymptotic expansion is given for the probability of the sum of the first n (integral) values assumed by the chain under a set of conditions. The expansion is the same as for the sum of independent, identically distributed, integral random variables. *K. L. Chung* (Syracuse, N.Y.)

2781:

Dugue, D. Sur la convergence stochastique au sens de Cesaro et sur des différences importantes entre la convergence presque certaine et les convergences en probabilité et presque completés. Sankhyā **18** (1957), 127–138.

Let $x_1, x_2, \cdots$ be a sequence of random variables and let $M_n=\max_{1\le i\le n} x_i$, $m_n=\min_{1\le i\le n} x_i$, and $\bar{x}_n=(x_1+\cdots+x_n)/n$. The author investigates relations between several kinds of convergence of $\bar{x}_n$ and the corresponding convergence of M_n/n and m_n/n as $n\to\infty$. [See also Dugué, Bull. Inst. Internat. Statist. **24** (1954), 2ème livraison, 60–71; MR **16**, 941]. In particular, he shows the following, with the aid of an inequality due to P. Lévy. If $x_1, x_2, \cdots$ are independent, the convergence in probability (complete convergence) of $\bar{x}_n$ to a constant implies the convergence in probability (complete convergence) of M_n/n and m_n/n to 0. This is a slight generalization of a well-known theorem of Kolmogoroff and of a theorem of Erdös [Ann. Math. Statist. **20** (1949), 286–291; Trans. Amer. Math. Soc. **67** (1949), 51–56; MR **11**, 40, 375]. Using this theorem, the author shows through examples that the convergence in probability (complete convergence) of a sequence of independent random variables $x_1, x_2, \cdots$ to zero does not necessarily imply convergence in probability (complete convergence) of $\bar{x}_n$. It follows that these two kinds of convergence cannot be founded on a normed topology. *L. Schmetterer* (Berkeley, Calif.)

2782:

Blum, J. R.; Chernoff, H.; Rosenblatt, M.; and Teicher, H. Central limit theorems for interchangeable processes. Canad. J. Math. **10** (1958), 222–229.

Let $\{X_i\}$ $(i=1, 2, \cdots)$ be an interchangeable process (i.e., one for which the distribution of $(X_{i_1}, X_{i_2}, \cdots, X_{i_n})$ depends only on n whenever the integers $i_1, i_2, \cdots, i_n$ are distinct) with mean zero and variance one. Let $S_n=\sum_{i=1}^n X_i$. Using the canonical representation of DeFinetti [Ann. Inst. H. Poincaré **7** (1937), 1–68], the authors find that a necessary and sufficient condition that $n^{-\frac{1}{2}}S_n$ be asymptotically normal, with mean zero and variance one, is that (1) X_1 and X_2 be uncorrelated and (2) X_1^2 and X_2^2 be uncorrelated. This theorem is extended in several ways to doubly infinite sequences $\{X_{ni}\}$, where for each n the random variables X_{ni} form an interchangeable process, with mean zero, variance one and finite absolute third moment. If $S_n=\sum_{i=1}^n X_{ni}$, then one set of conditions to insure the asymptotic normality of $n^{-\frac{1}{2}}S_n$ is that (1) $E\{X_{n1}X_{n2}\}=o(n)$, (2) $E\{X_{n1}^2-X_{n2}^2\}-1=o(1)$, and (3) $E\{|X_{n1}|^3\}=o(n^{\frac{1}{2}})$.

T. S. Ferguson (Los Angeles, Calif.)

2783:

Chover, J. Conditions on the realization of prediction by measures. Duke Math. J. **25** (1958), 305–310.

Let x_t, $-\infty<t<\infty$, be a second order stationary stochastic process. Assume $E\{x_t\}\equiv 0$. Let $\rho(t-s)=E\{x_s, x_t\}$, so that $\rho(t)$ can be written $\int_{-\infty}^{\infty} e^{it\lambda}dF(\lambda)$. Assume $dF(\lambda)=S(\lambda)d\lambda$, where $\int_{-\infty}^{\infty} |\ln S(\lambda)|/(1+\lambda^2)\, d\lambda<\infty$. For $-\infty\leqq a\leqq t<\infty$, $P_a{}^t$ will denote the orthogonal projection onto the closed linear manifold spanned by $\{x_s|a\leqq s\leqq t, -\infty<s\}$. In this paper, necessary conditions are found that $P_a{}^t x_c$ be realizable as a stochastic integral over the past, with respect to a measure. To be precise, Theorem: Let $-\infty\leqq a<b<c<\infty$. Suppose that for each t with $b<t<c$, there is a bounded complex Radon measure m^t on $[a, t]$ such that the stochastic integral $\int x_s dm^t(s)$ is equal to $P_a{}^t x_c$. Then none of the following conditions can hold: (I) $d\rho(0)/d\lambda$ exists and equals 0; (II) $D^+\rho(0)$ and $D^-\rho(0)$ exist, and $m^t(\{t\})=0$ for $b<t<c$; (III) $D^+\rho(0)$ and $D^-\rho(0)$ exist, and $\lim_{s\uparrow t} \|x_t-P_a{}^s x_t\|/(t-s)=0$, for $b<t<c$. The author remarks that condition (III) can frequently be checked, at least in the case $a=-\infty$. The proof proceeds by first showing that the function $f(t)=\|P_a{}^t x_c\|^2$ is (independently of the hypothesis) continuous for $a<t\leqq c$. Now, any of the conditions (I), (II), (III) would imply $D^-f(t)$ exists and equals 0 for $b<t<c$. Then $f(t)$ is a constant for $b<t\leqq c$, contradicting $\int |\ln S(\lambda)|/(1+\lambda^2)\, d\lambda<\infty$. *J. Feldman* (Berkeley, Calif.)

2784:

Bellman, Richard. On the representation of the solution of a class of stochastic differential equations. Proc. Amer. Math. Soc. **9** (1958), 326–327.

The author studies the non-linear differential equation

$$\frac{du}{dt}=g(u)+r(t),\ u(0)=c,$$

where g is a strictly convex twice differentiable function and $r(t)$ is a given stochastic process. It is shown that the distribution function of the solution u can be represented as

$$P[u\geqq x]=\max_v P[U(v; t)\geqq x],$$

where the maximization is over all stochastic processes $v(t)$, and U is the solution of the linear equation

$$\frac{dU}{dt}=g(v)+(U-v)g'(v)+r(t).$$

U. Grenander (Stockholm)

2785:

Dupač, Václav. Notes on stochastic approximation methods. Czechoslovak Math. J. **8(83)** (1958), 139–149. (Russian summary)

Asymptotic properties of the Robbins-Monro and the Kiefer-Wolfowitz stochastic approximation methods are studied under the assumption that the solution lies in an a priori known finite interval. It is shown that in this case the conditions under which the approximation procedure

has satisfactory asymptotic properties can be considerably weakened. A stochastic approximation method is considered for solving systems of linear equations with a symmetric matrix of coefficients. *J. Janko* (Prague)

2786:

Fisz, M. A central limit theorem for some stochastic processes. Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys. **6** (1958), 437–443.

The author's principal theorem is: Let $\{Y_k(t),\ 0\leqq t\leqq 1\}$ $(k=1, 2, \cdots)$ be a sequence of real, separable, independent and identically distributed stochastic processes. Let $F(t)=EY_k(t)$ $(0\leqq t\leqq 1)$, $X_n(t)=n^{-1}\sum_{k=1}^n Y_k(t)$, $\xi_n(t)=n^{\frac{1}{2}}[X_n(t)-F(t)]$. Denote by $D[0, 1]$ the complete, separable metric space of real functions defined on the interval $[0, 1]$, having both right- and left-hand limits at each point t and continuous from the left (at $t=0$ from the right) with the metric d introduced by Prohorov [Yu. V. Prohorov, Teor. Veroyatnost. i Primenen. 1 (1956), 177–238; MR **18**, 943; p. 228]. Assume that:

α) For arbitrary t_1, t_2, where $0<t_1\leqq t_2<1$,

$$EY_k(t_1)Y_k(t_2)-F(t_1)F(t_2)=u(t_1)v(t_2),$$

where $u(t)/v(t)$ is a continuous and monotonically increasing function; $E[Y_k(t_2)-Y_k(t_1)]^2\leqq C_1(t_2-t_1)$, where C_1 does not depend on t.

β) For arbitrary $t_1, t_2, t_3\in[0, 1]$, where $t_1<t_2<t_3$, the relation

$$E|Y_k(t_2)-Y_k(t_1)|^j\,|Y_k(t_3)-Y_k(t_2)|^m \leqq C_2(t_2-t_1{}^{j/2})(t_3-t_2)^{m/2}$$

holds for $j, m=1, 2$; and C_2 does not depend on t.

γ) The probability that $Y_k(t)$ is continuous from the left at those points t at which the left-hand limit exists (at the point $t=0$ from the right if $Y_k(+0)$ exists) is equal to 1.

Then, with probability 1, the realizations of $\xi_n(t)$ belong to $D[0, 1]$, and $P^{\xi_n}\Rightarrow P^{\xi_0}$, where $\{\xi_0(t),\ 0\leqq t\leqq 1\}$ is a real separable, Gaussian stochastic process, with $E\xi_0(t)=0$ $(0\leqq t\leqq 1)$, $E\xi_0(t_1)\xi_0(t_2)=u(t_1)v(t_2)$ $(0\leqq t_1\leqq t_2\leqq 1)$, and where P^{ξ_n} and P^{ξ_0} are probability measures generated in the space $D[0, 1]$ by the finite dimensional distributions of $\xi_n(t)$ and $\xi_0(t)$, respectively. *J. Wolfowitz* (Ithaca, N.Y.)

2787:

Lévy, Paul. Sur une classe de courbes de l'espace de Hilbert et sur une équation intégrale non linéaire. Ann. Sci. Ecole Norm. Sup. (3) **73** (1956), 121–156.

Die zum Teil schon angekündigten Ergebnisse [C. R. Acad. Sci. Paris **242** (1956), 1252–1255; MR **17**, 978] betreffen die Bestimmung des Kernes $F(t, u)$ eines Gaussischen Prozesses $X(t)$ mit $E(X(t))=0$ durch dessen Kovarianzfunktion $\Gamma(t_1, t_2)=E(X(t_1)X(t_2))$. Die Darstellung erwähnt nur die wahrscheinlichkeitstheoretische Deutung und benutzt die Sprache des Hilbertschen Raumes. Betrachtet wird dann also eine durch $\Gamma(t_1, t_2)$ bis auf Drehung festgelegte Kurve, deren besondere geometrische Eigenschaften, wie z.B. Projektionsmöglichkeit auf eine Gerade mit gegebenem Zeitablauf auf derselben, ausführlich untersucht werden. Dabei ergeben sich auch neue Charakterisierungen des kanonischen Kernes, einfache hinreichende Bedingungen für Kerne, kanonisch zu sein, usw. Für gewisse Kovarianzfunktionen Γ erfolgt dann die Bestimmung von $F(t, u)$ in drei Schritten: (1)

$$\sigma^2(t)=\lim_{\tau\to 0}\frac{1}{|\tau|}\{\Gamma(t, t)-2\Gamma(t, t+\tau)+\Gamma(t+\tau, t+\tau)\}$$

als existent $\neq 0$ vorausgesetzt, ergibt später $F(t, t)$. (2) Mit $\gamma(t_1, t_2)=\partial^2\Gamma(t_1, t_2)/\partial t_1\partial t_2$ Lösung der für jedes t als eindeutig lösbar nachgewiesenen Fredholmschen Integralgleichung

$$\gamma(t, x)=G(t, x)\sigma^2(x)+\textstyle\int_0^t G(t, u)\gamma(u, x)dx \quad (0<x<t).$$

(3) Bestimmung des lösenden Kernes $R(t, u)$ der Volterraschen Integralgleichung $V(t)-\int_0^t G(t, u)V(u)du=Z(t)$. Dann ist $F(t, u)=\sigma(u)[1-\int_u^t R(v, u)dv]$ die bei Forderung der Stetigkeit bis auf das Vorzeichen eindeutig bestimmte Lösung der Integralgleichung. Viele Beispiele, z.B. Polynomkerne und der Wienersche Prozeß, erläutern die umfangreichen allgemeinen Überlegungen, deren Fülle an Einzelheiten und Ergebnissen man aus der Arbeit selbst entnehmen muß. *D. Morgenstern* (Zbl **71**, 350)

2788:

Lévy, Paul. Processus strictement markoviens. C. R. Acad. Sci. Paris **246** (1958), 1490–1492.

A random function $X(t)$ is called weakly Markovian if, for every increasing sequence $\{t_n\}$, the sequence $\{X(t_n)\}$ is a simple Markov chain. It is strictly Markovian if, for all t, the conditional probability of any future event, given $X(t)=x$, is not altered by knowledge of past values of the function. It is almost strictly Markovian if the above property holds only for almost all values of $X(t)$. The complete definition of $X(t)$ requires its specification as a sure function of t and $\omega\in\Omega$, where Ω is the sample space.

On the basis of the above concepts the author poses and sketches partial solutions of two problems: (1) Given the function P, deduce the complete definition of some Markov process having P for its transition probability; (2) define a strictly Markovian function having a given transition probability. *H. P. Kramer* (Murray Hill, N.J.)

2789:

Austin, D. G.; Blumenthal, R. M.; and Chacon, R. V. On continuity of transition functions. Duke Math. J. **25** (1958), 533–541.

Continuing work of J. L. Doob [Trans. Amer. Math. Soc. **52** (1942), 37–64; MR **4**, 17], A. N. Kolmogorov [Moskov. Gos. Univ. Uč. Zap. **148**, Mat. **4** (1951), 53–59; MR **14**, 295], and D. G. Kendall [Trans. Amer. Math. Soc. **78** (1955), 529–540; MR **16**, 725], the authors consider the smoothness of Markov transition functions $P_t(a, B)$ in their dependence on $t\geqq 0$ for fixed (points) a and (measurable sets) B. A sample result is that, for fixed a, $P_t(a, B)$ is continuous from the right in $t\geqq 0$ for each (measurable) B if and only if it possesses a density function, measurable in the pair (t, b).

H. P. McKean Jr. (Cambridge, Mass.)

2790:

Hajnal, J. Weak ergodicity in non-homogeneous Markov chains. Proc. Cambridge Philos. Soc. **54** (1958), 233–246.

The paper deals with weak ergodicity, i.e., the tendency for a chain to "forget" the distant past. This may occur in non-homogeneous chains, even if the probabilities of being in a given state do not tend to a limit as the number of trials increases. Investigation of the general conditions of weak ergodicity leads to the definition of a special class of regular matrices. Conditions for chains involving regular matrices are also given (§§ 2.3). § 4 discusses similarities in asymptotic behavior between chains whose transition matrices differ only by small amounts. (Author's abstract) *K. L. Chung* (Syracuse, N.Y.)

2791:
Šur, M. G. Ergodic properties of invariant Markov chains on homogeneous spaces. Teor. Veroyatnost. i Primenen. **3** (1958), 137–152. (Russian. English summary)

The author considers discrete time Markov chains defined on a homogeneous measurable space $(X, G, \mathfrak{B})$. Here G is a transitive group of transformations of the state space X and $\mathfrak{B}$ is a σ-algebra of subsets of X. The chain is called invariant if the transition function $P(x, E)$ has the property that $P(x, E)=P(gx, gE)$ for x in X and g in G. It is strictly regular on $H\subseteq X$ if $H=\bigcup E_\alpha$ (E_α pairwise disjoint), $E_\alpha=\bigcup_{i=1}^{d_\alpha} E_{\alpha,i}$ ($E_{\alpha,i}$ for fixed α pairwise disjoint) and, for $x\in E_{\alpha,i}$, $E\in\mathfrak{B}$, $\lim P^{(nd_\alpha+j-i)}(x, E)=\varphi_{\alpha,j}(E)$ exists and is a probability measure on $\mathfrak{B}$. Applying the results of Doeblin [Ann. Ecole. Norm. (3) **57** (1940), 61–111; MR **3**, 3] for abstract space discrete time chains, the author obtains necessary and sufficient conditions for an invariant chain to be strictly regular.

J. L. Snell (Palo Alto, Calif.)

2792:
Gihman, I. I. A limit theorem for the number of maxima in a sequence of random variables in a Markov chain. Teor. Veroyatnost. i Primenen. **3** (1958), 166–172. (Russian. English summary)

The author proves a limit theorem concerning the number of maxima in a sequence of random variables forming a Markov chain and converging to a Markov process of the diffusion type. It is assumed that the transition probabilities $P(t, x, \tau, A)$ and the generated operator of the limiting Markov process satisfy conditions given by Feller [Math. Ann. **113** (1936), 113–160].

H. P. Edmundson (Santa Monica, Calif.)

2793:
Bharucha-Reid, A. T.; and Rubin, Herman. Generating functions and the semigroup theory of branching Markov processes. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 1057–1060.

The authors state: "In this note we consider an approach to the study of Markov processes with a denumerable state space which is based on the properties of the semi-group of operators associated with the generating function of the probabilities defining a particular process." A number of theorems are stated without proof; citation of the theorems would be tantamount to a repetition of the paper.

J. Wolfowitz (Ithaca, N.Y.)

2794:
Gillis, J. Centrally biased discrete random walk. Quart. J. Math. Oxford Ser. (2) **7** (1956), 144–152.

Consider a random walk on a d-dimensional lattice, where $m=(m_1, \cdots, m_d)$ is a general point. Let $P_i(m)$ and $Q_i(m)$ be the probabilities of moving one unit parallel to the ith axis in the positive or negative direction respectively. Assume $P_i(m)=(1-\varepsilon/m_i)/2d$, $Q_i(m)=(1+\varepsilon/m_i)/2d$ if $m_i\neq 0$, and $P_i=Q_i=(2d)^{-1}$ if $m_i=0$, with $|\varepsilon|<1$. Theorem: The random walk is or is not recurrent according as $\varepsilon>(d-2)/2d$ or $\varepsilon<(d-2)/2d$. In general it is not known whether the walk is recurrent if the equality holds, although this is known to be true in 1 and 2 dimensions. The method makes use of the fact that recurrence is equivalent to divergence of $\Sigma_N \gamma_N^{(d)}$, where $\gamma_N^{(d)}$ is the probability that a point starting at 0 is at 0 after $2N$ steps. To estimate $\gamma_N^{(d)}$, the author uses the relation

$$\gamma_N^{(d)}=d^{-2N}\sum_{n_1+\cdots+n_d=N}\frac{(2N)!}{(2n_1)!\cdots(2n_d)!}\gamma_{n_1}^{(1)}\cdots\gamma_{n_d}^{(1)},$$

in conjunction with estimates of $\gamma_N^{(1)}$.

T. E. Harris (Santa Monica, Calif.)

2795:
Trotter, H. F. A property of Brownian motion paths. Illinois J. Math. **2** (1958), 425–433.

"Let $x(t, \omega)$ be a separable one-dimensional Brownian motion process with $x(0, \omega)=0$. We suppose that, if necessary, a set of sample points of measure zero has been discarded, so that all the sample functions of the process are continuous. Then, for every t and ω, a measure $\mu(\cdot, t, \omega)$ is defined by taking $\mu(A, t, \omega)$ to be the Lebesgue measure of the set $\{\tau\colon 0\leq\tau\leq t,\ x(\tau, \omega)\in A\}$. In this paper we prove the following result. Theorem: With probability one, $\mu(\cdot, t, \omega)$ has a continuous density function. That is, for almost all ω, there exists a function $f(x, t, \omega)$ which is continuous in x and t such that $\mu(A, t, \omega)=\int_A f(x, t, \omega)dx$ for every Borel set A.

Explicit bounds for the moduli of continuity of f are given." (Author's summary)

J. Wolfowitz (Ithaca, N.Y.)

2796:
Luchak, George. The continuous time solution of the equations of the single channel queue with a general class of service-time distributions by the method of generating functions. J. Roy. Statist. Soc. Ser. B **20** (1958), 176–181.

Customers arrive at random at a single channel queue. There is a probability c_m that a customer will require m "phases" of service. Each phase has negative exponential service time distribution, and the phase services are independent. The author gives series solutions for $P_0(t)$, the probability that the server is free, and $m_1(t)$ and $m_2(t)$, the first two moments of the queue length (in phases), at time t when there were initially a phases at $t=0$. Similar results are provided for the distribution of the busy period.

D. V. Lindley (Cambridge, England)

2797:
Conolly, B. W. A difference equation technique applied to the simple queue. J. Roy. Statist. Soc. Ser. B **20** (1958), 165–167.

The Laplace transform of $p_n(t)$, the probability that there are n customers in the queue at time t, given that there were a at $t=0$, is found for the system $M/M/1$.

D. V. Lindley (Cambridge, England)

2798:
Conolly, B. W. A difference equation technique applied to the simple queue with arbitrary arrival interval distribution. J. Roy. Statist. Soc. Ser. B **20** (1958), 168–175.

The Laplace transform of $p_n(t)$, the probability that there are n customers in the queue at time t, given that there was only one customer in the queue at $t=0$ and he had just arrived, is found for the system $GI/M/1$. Assuming the existence of $\lim_{t\to\infty} p_n(t)$, its value is found. The novelty of this, and the previous paper, lies in the algebraic manipulation of the standard equations.

D. V. Lindley (Cambridge, England)

2799:
Takács, L. On limiting distributions concerning a sojourn time problem. Acta Math. Acad. Sci. Hungar. **8** (1957), 279–294.

This is a continuation of the author's earlier work [same Acta **8** (1957), 169–191; MR **19**, 467], where he derived the asymptotic distribution of the sojourn time $\beta(t)$ [for the notation see the review quoted] under the assumption that the distributions $A(x)$ and $B(x)$ have both finite second moments σ_α and σ_β respectively. In the present paper it is assumed that $A(x)$ [(Bx)] satisfy either $\sigma_\alpha<\infty$

$[\sigma_\beta<\infty]$ or $\lim_{x\to\infty}\{1-A(x)\}x^{\gamma_1}=A$ $[\lim_{x\to\infty}\{1-B(x)\}x^{\gamma_2}=B]$ where $0<\gamma_1<2$, $0<\gamma_2<2$, $A>0$, $B>0$. Asymptotic distributions of $\beta(t)$ are derived for the nine combinations of the conditions $\sigma_\alpha<\infty$, $1<\gamma_1<2$, $0<\gamma_1<1$ with the conditions $\sigma_\beta<\infty$, $1<\gamma_2<2$, $0<\gamma_2<1$.

E. Lukacs (Washington, D.C.)

2800:

Takács, Lajos. Sojourn time problems. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **7** (1957), 371–395. (Hungarian)

This is an exposition, in Hungarian, of results which the author published previously in English [Acta Math. Acad. Sci. Hungar. **8** (1957), 169–191; Teor. Veroyatnost. i. Primenen. 2(1957), 92–105; MR **19**, 467, 692; and #2799 above].

E. Lukacs (Washington, D.C.)

2801:

Takács, L. On a probability problem concerning telephone traffic. Acta Math. Acad. Sci. Hungar. **8** (1957), 319–324.

The author gives a simpler proof for his generalization of Erlang's formula [same Acta **7** (1956), 419–433; MR **19**, 623]. It is noted that the same result was also obtained earlier by C. Palm [Ericson Technics no. **44** (1943), MR **6**, 160] and by F. Pollaczek [C.R. Acad. Sci. Paris **236** (1953), 1469–1470; MR **14**, 773]. The author finally shows the connection with a servicing problem.

R. G. Laha (Washington, D.C.)

2802:

Takács, L. On a queueing problem concerning telephone traffic. Acta Math. Acad. Sci. Hungar. **8** (1957), 325–335.

The author considers a telephone traffic problem and assumes that the inter-arrival times of incoming calls are independently and identically distributed. Their common distribution is arbitrary, while the service times are independently distributed, each with an exponential distribution. Under certain conditions, which are too complicated to be stated here, he derives explicit expressions for the probabilities P_k that the sum of the numbers of calls served and waiting equals k. He also determines the distribution of waiting times for an arbitrary call. It is noted that the result that the probabilities of the number of calls in the waiting line form a geometric series was obtained earlier by D. G. Kendall [Ann. Math. Stat. **24** (1953), 338–354; MR **15**, 44].

R. G. Laha (Washington, D.C.)

2803:

Takács, Lajos. Some probability questions in the theory of telephone traffic. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **8** (1958), 151–210. (Hungarian)

This is a comprehensive survey of queuing problems in telephone traffic. Let $0<\tau_1<\cdots<\tau_n<\cdots$ be the arrival times of incoming calls and suppose that the inter-arrival times $\tau_{n+1}-\tau_n$ are independently and identically distributed positive random variables and that the $\{\tau_n\}$ form a recurrent process. Moreover, suppose that the holding times are exponentially and independently distributed random variables which are also independent of the arrival times. It is assumed that m channels are available. If a channel is free at the time a call arrives the connection is made; if all m lines are busy and if the number of waiting calls does not exceed w, then the incoming call is held until a line becomes available. A call which arrives at a moment when all m lines are busy and more than w calls are waiting is lost. Three models are studied: In model (I) $w=0$, in model (II) $w=\infty$ while in model (III) $0<w<\infty$.

Let $\eta(t)$ be the sum of the number of busy lines and the number of waiting calls and write $\eta_n=\eta(\tau_n-0)$. Denote the length of the time interval between t and the arrival of the next incoming call by $\zeta(t)$. The author studies the limiting distribution of the random variables η_n and $\eta(t)$ as n and t respectively tend to infinity. The limit of the conditional distribution $P[\zeta(t)\leqq x|\eta(t)=k]$ is determined under the assumption that the distribution of inter-arrival times is not a lattice distribution and that it has a finite first moment. A derivation of Palm's formula for the probability that a call is lost is given, and the case $m=\infty$ (then one is necessarily in case (I)) is studied in detail.

E. Lukacs (Washington, D.C.)

2804:

Takács, L. On a coincidence problem concerning telephone traffic. Acta Math. Acad. Sci. Hungar. **9** (1958), 45–81.

This is a detailed discussion of a telephone system with an infinite number of available channels under various assumptions concerning the distributions of the arrival times and of the holding times. This paper, as well as the paper reviewed above, contains many results (or extensions of results) which were given by the author in previous papers [in particular: Acta Math. Acad. Sci. Hung. **7** (1956), 419–433; MR **19**, 623; Teor. Veroyatnost i Primenen. **2** (1957), 92–105; MR **19**, 692; and #2801, #2802 above] and which are also listed in his extensive bibliographies.

E. Lukacs (Washington, D.C.)

2805:

Zolotarev, V. M. More exact statements of several theorems in the theory of branching processes. Teor. Veroyatnost. i Primenen. **2** (1957), 256–266. (Russian. English summary)

The author studies stationary, continuous parameter, regular branching processes with particles of one type. Such a process is called regular if at every positive time point the number of descendents of a single particle is finite, with probability one. He obtains a number of results which complement the results of Kolmogorov [Proc. Tomsk **2** (1938), 1–12], Yaglom [Dokl. Akad. Nauk SSSR (N.S.) **56** (1947), 795–798; MR **9**, 149] and Sevastyanov [Uspehi Mat. Nauk (N.S.) **6** (1951), no. 6(46), 47–99; MR **13**, 763]. It would be impractical to cite all these results in detail here, and we content ourselves with a typical one: Theorem 6 states that, if the first factorial moment of the probability generating function is negative (which, incidentally, implies that the process will terminate within finite time, with probability one), the distribution of the number of descendents at time t, conditional upon this number's being positive, approaches a limit whose probability generating function has a form described in the paper.

J. Wolfowitz (Ithaca, N.Y.)

2806:

Hida, Takeyuki. On the uniform continuity of Wiener process with a multidimensional parameter. Nagoya Math. J. 13 (1958), 53–61.

For each point A in Euclidean N-space let $X(A)$ be a random variable, and suppose that the $X(A)$ process is Brownian motion, so that the process is Gaussian, $X(0)=0$, $X(A)-X(B)$ has mean 0 and variance $r(A,B)$, the distance between A and B. The author proves the following theorems. Let

$$\phi_c(r)=\{r[2N|\log r|+c \log |\log r|]\}^{\frac{1}{2}}.$$

(1) If $c>8N+1$, and if $r(0, A)\leq 1$, then for almost every sample function there exists a strictly positive ρ, depending on the sample function, such that if $r=r(A, B)<\rho$, it follows that $|X(A)-X(B)|\leq\phi_c(r)$. (2) If $c<1$, such a ρ exists for almost no sample function.

These theorems strengthen theorems of Lévy (who considered multiples of $\phi_0(r)$) [Processus stochastiques et mouvement brownien, Gauthier-Villars, Paris, 1948; MR **10**, 551] by extending theorems of Sirao [J. Math. Soc. Japan **6** (1954), 332–335; MR **16**, 725] from 1 to N dimensions. *J. L. Doob* (Urbana, Ill.)

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See also 3036.

2807:

Quandt, R. E. Probabilistic errors in the Leontief system. Naval Res. Logist. Quart. **5** (1958), 155–170.

Assume that the matrix, A, of input-output coefficients can be regarded as a random variable while final demands, Y, are known without error. Then the vector X of outputs, defined as the solution of the equation $(I-A)X=Y$, is a random variable. If the elements of A are independent random variables, then so are the elements of X, and the variance of any element is determined by Y and the variances and covariances of elements of a row of $(I-A)^{-1}$. Each element of $(I-A)^{-1}$ is a ratio, and the author first derives formulae for the variance of a ratio x/y and the covariance of two ratios of the forms, x/y, z/y; the formulae are approximations which disregard powers of deviations from means above the second. He then uses these results to derive the variance and covariance of the elements of a row of $(I-A)^{-1}$ when the matrix has order 2. The method generalizes, but the algebra rapidly becomes excessive. {The formulas given contain some minor errors. In (4.9), the definition of K should be 1 less than that given. The number 1 should be omitted from all elements of (4.14).} *K. J. Arrow* (Stanford, Calif.)

2808:

Ikeda, Nobuyuki; and Morimoto, Haruki. Notes on some relations between the distributions and sufficient statistics. Mem. Fac. Sci. Kyusyu Univ. Ser. A **12** (1958), 12–21; correction, **13** (1959), no. 1, unbound insert.

The paper gives a characterization of exponential (or Koopman) families of distributions in terms of properties of the k-fold minimal sufficient statistic for samples of size n. Not only regularity properties are invoked but also very special relations between the minimal sufficient statistic for samples of size n and $m+n$. Some special kinds of exponential families are also briefly explored. *L. J. Savage* (Rome)

2809:

Thionet, Pierre. Sur les rapports entre divers concepts d'information. C. R. Acad. Sci. Paris **246** (1958), 223–224.

The connections between the Fisher and Shannon-Lindley informations are discussed in the case of sampling from a finite population. *D. V. Lindley* (Cambridge, England)

2810:

Weiss, Lionel. The convergence of certain functions of sample spacings. Ann. Math. Statist. **28** (1957), 778–782.

For $u_i>0$ $(1\leq i\leq k)$, suppose $g(u_1, \cdots, u_k)$ is continuous, homogeneous of order r, nondecreasing in each argument, positive, and approaches ∞ uniformly with $\sum u_i$. Let ρ be bounded and non-negative on [0, 1]. Let $X_1, \cdots, X_n$ be independent with common density f and common d.f. F, where $F(0)=0$ and $F(1)=1$. Let $Y_1, \cdots, Y_n$ be the corresponding order statistics, and let $R_n(t)$ denote the proportion of the values $\rho(j/n)g(Y_{j+1}-Y_j, Y_{j+2}-Y_{j+1}, \cdots, Y_{j+k}-Y_{j+k-1})$, $1\leq j\leq n-k$, which are less than t/n^r. The author proves that, as $n\to\infty$, $R_n(t)$ approaches stochastically, uniformly in t, a limit $s(t)$ which he computes. The result, which generalizes the author's result in same Ann. **26** (1955), 532–536 [MR **17**, 48], can be used to test the hypothesis that a given d.f. belongs to a family which is specified except for scale and location parameters. *J. Kiefer* (Oxford)

2811:

Weiss, Lionel. The asymptotic power of certain tests of fit based on sample spacings. Ann. Math. Statist. **28** (1957), 783–786.

Let $X_1, X_2, \cdots$ be independent with common density f on [0, 1], where f has finitely many discontinuities and $0<A<f(x)<B<\infty$ for $0\leq x\leq 1$ and some A and B. Let $Y_0=0$, $Y_{n+1}=1$, and let $Y_1, \cdots, Y_n$ be the ordered X_i, $1\leq i\leq n$. Let $T_i=Y_i-Y_{i-1}$, $1\leq i\leq n+1$. For $r>0$, let $V(n)=\sum_1^{n+1} T_i^r$. The author proves that $V(n)$ is asymptotically normal as $n\to\infty$, the mean and variance being explicitly given. This generalizes a result of Darling [same Ann. **24** (1953), 239–253; MR **15**, 444] for the case $f(x)=1$. The result can be used to compute the asymptotic power of the test of size α of the hypothesis that $f(x)=1$, whose critical region is $V(n)>C_n(\alpha)$. *J. Kiefer* (Oxford)

2812:

Banerjee, D. P. On the exact distribution of a test in multivariate analysis. J. Roy. Statist. Soc. Ser. B **20** (1958), 108–110.

M. G. Kendall [The advanced theory of statistics, Griffin, London, 1946; MR **8**, 473; § 28.16; the reference given by the author is wrong] has described a criterion L, due to Wilks, to test the equality of the means of several multivariate normal distributions with common, but unknown, dispersion matrix. The author finds the probability density of L, on the null hypothesis, as a power series in L. *D. V. Lindley* (Cambridge, England)

2813:

Kudô, Akio. The extreme value in a multivariate normal sample. Mem. Fac. Sci. Kyusyu Univ. Ser. A **11** (1957), 143–156.

The problem is to provide a test, to be applied to a multivariate normal sample, which will distinguish an observation which is so extreme that it may be considered to be discordant and hence may be rejected from the sample. The proposed test is based on the deviate function

$$(1)\qquad \max_{\nu=1,2,\cdots,n} \sum_{i,j}^{k} \sigma^{ij}(x_i^{(\nu)}-\bar{x}_i)(x_j^{(\nu)}-\bar{x}_j)$$

for a multivariate normal sample $x^{(\nu)}=(x_1^{(\nu)}, x_2^{(\nu)}, \cdots, x_k^{(\nu)})$ and known variance-covariance matrix σ. Let $m^{(\nu)}=(m_1^{(\nu)}, m_2^{(\nu)}, \cdots, m_k^{(\nu)})$, be the vector of means corresponding to observation ν and $H_0(m^{(\nu)}=m)$ the null hypothesis. Alternative hypotheses H_ν specify that $x^{(\nu)}$ (only) has a mean vector different from the others. Let D_ν $(\nu=0, 1, \cdots, n)$ be the decision to accept H_ν subject to certain criteria. It is shown that if M is a value of ν which satisfies (1), and if λ_p is so chosen that the probability of

correct decision D_0 when H_0 is correct is $1-p$, then we accept D_M if

$$(2)\qquad \sum_{i,j} \sigma^{ij}(x_i^{(M)}-\bar{x}_i)(x_j^{(M)}-\bar{x}_j)\geqq\lambda_p.$$

Otherwise we accept D_0.

There are generalizations but the author does not yet give a derivation for the case in which the variance-covariance matrix is replaced by its estimate.

P. S. Dwyer (Ann Arbor, Mich.)

2814:

Govidarajulu, Z. A note on the correlation coefficient of the bivariate gamma type distribution. J. Madras Univ. Sect. B. **26** (1956), 639–642.

The author evaluates the correlation coefficient between X and Y, when the joint d.f. is bivariate gamma. Since the m.g.f. is $G(\tau, s)=[(1-\tau)(1-s)-\tau s\rho^2]^{-p}$ [Kibble, Sankhyā **5** (1940), 137–150; MR **4**, 103], it is easily verified that $\rho_{xy}=\rho^2$. *I. Olkin* (Stanford, Calif.)

2815:

Geisser, Seymour; and Greenhouse, Samuel W. An extension of Box's results on the use of the F distribution in multivariate analysis. Ann. Math. Statist. **29** (1958), 885–891.

2816:

Sandelius, Martin. On the estimation of the standard deviation of a normal distribution from a pair of percentiles. Skand. Aktuarietidskr. **40** (1957), 85–88.

2817:

Geffroy, Jean. Stabilité presque complète des valeurs extrêmes d'un échantillon et convergence presque complète du milieu vers une limite certaine. C. R. Acad. Sci. Paris **246** (1958), 224–226.

This note, as well as the one reviewed below, is a continuation of the authors' previous studies [same C. R. **245** (1957), 1215–1217, 1291–1293; MR **19**, 690]. "We will show in this note how the almost complete convergence of the mid-range of a random sample is related to the almost complete stability of the extreme values. The latter property is studied in more detail." (The author's summary.) *R. Pyke* (New York, N.Y.)

2818:

Geffroy, Jean. Etude de la stabilité presque certaine des valeurs extrêmes d'un échantillon et de la convergence presque certaine de son milieu. C. R. Acad. Sci. Paris **246** (1958), 1154–1156.

"The object of this note is to show that there exists a family of probability distributions for which the maximum value of a random sample is stable almost surely. The normal distribution is a member of this family". (The authors' summary.) The proposed family consists of all distribution functions F satisfying

$$\lim_{x\to+\infty} G(x-\varepsilon)/G(x)\log G(x)=-\infty$$

for all $\varepsilon>0$ where $G=1-F$.

R. Pyke (New York, N.Y.)

2819:

Walsh, John E. Efficient small sample nonparametric median tests with bounded significance levels. Ann. Inst. Statist. Math., Tokyo **9** (1958), 185–199.

Let θ be the median of a continuous distribution and $x_1, x_2, \cdots, x_n$ the order statistics of a sample of size n. Non-parametric tests of hypotheses and confidence intervals for values of θ based upon statistics such as $\min[\frac{1}{2}(x_1+x_{1+i}), x_2]$, $\max[\frac{1}{2}(x_n+x_{n-i}), x_{n-1}]$, $\frac{1}{2}(x_1+x_{1+i})$, and $\frac{1}{2}(x_n+x_{n-i})$ are discussed. Tables giving bounds for confidence coefficients and significance levels are presented for the proposed procedures. *F. C. Andrews* (Eugene, Ore.)

2820:

Josífko, Marcel. The characteristic function of the Kendall's rank correlation coefficient. Časopis Pěst. Mat. **83** (1958), 56–59. (Czech. Russian and English summaries)

The characteristic function of the random variable S in Kendall's coefficient of rank correlation $\tau=S:\binom{n}{2}$ in case of independence of the random variables X and Y is given. By using the characteristic function the cumulants and asymptotic distribution of τ are derived.

J. Janko (Prague)

2821:

Bankier, J. D.; and Walpole, R. E. Components of variance analysis for proportional frequencies. Ann. Math. Statist. **28** (1957), 742–753.

The authors consider the case of a 2-way classification with "cell repetition" in which the frequencies in the cells are proportional to the marginal frequencies. The analysis of variance for such a situation was previously considered by Snedecor, Cox, and H. F. Smith. In the present paper the authors give in more detail the theoretical background for the analysis for the case of a "Type I" model. They also generalize the results to cover 2-way cross classifications and 2-way nested classifications. Throughout standard least square approach is employed.

H. O. Hartley (Ames, Iowa)

2822:

Lehmann, E. L. A theory of some multiple decision problems. II. Ann. Math. Statist. **28** (1957), 547–572.

[Part I appears in Ann. Math. Statist. **28** (1957), 1–25; MR **18**, 955]. The author shows how a class of multiple decision problems may be generated by the simultaneous consideration of a number of two-decision problems. All of the problems considered in Part I correspond to the case in which the parameter space is partitioned into fixed subsets and one wishes to determine which of these subsets contains the true parameter point. The present paper considers the important class of problems where the statistician is permitted not to come to a definite conclusion regarding some questions; where the sharpness of his statements may depend on the experimental data; where, from a single set of observations, he wishes to answer a number of questions in sequence, and where the questions posed at a given stage depend, in turn, on the answers given at previous stages. In these cases decisions are again identified with subsets of the parameter space, but these subsets do not necessarily constitute a partition of the parameter space. The class of problems proved in Part I to be unbiased with uniformly minimum risk (in the sense of Part I) is extended in the direction in which the generated set of two-decision component problems can be thought of as existing in parallel. When the component problems arise in sequence, weaker results are indicated and the property of unbiasedness is related to a minimax property. *H. Raiffa* (Cambridge, Mass.)

2823:

Robson, D. S. Admissible and minimax integer-valued estimators of an integer-valued parameter. Ann. Math. Statist. **29** (1958), 801–812.

The author considers the case of a monotone likelihood

problem with convex loss function and finite action space. By using convexity he obtains a slightly smaller complete class than that obtained by S. Karlin and the reviewer. This class excludes all degenerate procedures. He shows that if the loss function is $W(\mu, \alpha)=|\alpha-\mu|^k$ for large k, all admissible procedures take only two actions, and if $W(\mu, \alpha)=|\alpha-\mu|$, all procedures are admissible. Minimax procedures are considered for these cases.

H. Rubin (Eugene, Ore.)

2824:

Johns, M. V., Jr. Non-parametric empirical Bayes procedures. Ann. Math. Statist. **28** (1957), 649–669.

Let x be an r-vector, y a scalar and let (x, y), (x_1, y_1), (x_2, y_2), $\cdots$ be i.i.d. with distribution P belonging to a class $\mathscr{P}$ and such that $Ey^2<\infty$. The theorems of greatest independent interest concern consistent estimation of $E[y|x]$ when $\mathscr{P}x^{-1}$ is dominated by (d) a discrete measure or (L) Lebesgue measure.

In (d) it is shown that the empirical averages,

$$\sum_1^n y_i I_{x_i}(x) / \sum_1^n I_{x_i}(x) \text{ (or 0 if } \sum_1^n I_{x_i}(x)=0),$$

converge to $E[y|x]$ in quadratic mean. The case (L) is reduced to similarity with (d) by a sequence of cubical partitions of r-space and the corresponding conclusion is obtained.

The principal empirical Bayes application is to the squared-deviation-loss prediction of $E[y|F]$ where, conditional on F, x is the set of order statistics of r independent F-distributed variables and y is a function of an $(r+1)$st. $E[E[y|F]|x]=E[y|x]$ insures the orthogonality of $E[y|F]-E[y|x]$ and $E[y|x]$, thus reducing this prediction problem to the estimation one. *J. Hannan* (East Lansing, Mich.)

2825:

Méric, Jean. Sur une méthode matricielle pour le calcul de la fonction O. C. du test binomial de Wald. C. R. Acad. Sci. Paris **246** (1958), 884–887.

Continuing the work of his previous paper [C. R. Acad. Sci. Paris **245** (1957), 1500–1502; MR **19**, 783], the author presents a somewhat more practical method for calculating the function mentioned in the title.

J. Kiefer (Oxford)

2826:

✱Gumbel, E. J. Statistics of extremes. Columbia University Press, New York, 1958. xx+375 pp. $15.00.

This book provides a detailed account of the theory of extreme values in independent samples from a fixed distribution, together with descriptions of some applications. The first chapter (41 pages) very briefly introduces some probability results needed in the subsequent work, defines the intensity function $\mu(x)=f(x)/(1-F(x))$, for a distribution function $F(x)$ with density $f(x)$, and the return period $(1-F(x))^{-1}$, and explains the use of probability paper. (Throughout the book much use is made of graphical methods.) The distributions of order statistics (mth largest value) and exceedances (the probability of exceeding the mth largest) are considered in the second chapter (33 pages). The latter provide the few distribution-free methods in the book. The third chapter (38 pages) is devoted to exact methods and, in particular, to a study of u_n, the characteristic largest value in a sample of n, defined by $F(u_n)=n^{-1}$, and the extremal intensity function $\alpha_n=\mu(u_n)$. This study is continued in chapter four (37 pages) for the exponential and Cauchy types. The next three chapters (150 pages in all) are based on the Fréchet-Fisher-Tippett asymptotic theory and a final chapter (35 pages) is devoted similarly to the range.

It is very much a specialist's book. The mathematician will find the style unsatisfactory; the engineer will probably be more disturbed by the mathematical appearance of the book than he need be. The theory, as the author freely admits, is not in a very elegant and complete state at the moment and this prevents the book from being as satisfactory as one might hope for. It is surprising, for example, that so little use is made of stochastic process theory. Nevertheless, its patient and exhaustive study of existing methods will be of value to workers who have to handle extreme values.

D. V. Lindley (Cambridge, England)

2827:

Grenander, Ulf. Bandwidth and variance in estimation of the spectrum. J. Roy. Statist. Soc. Ser. B **20** (1958), 152–157.

The author developed interesting results in an earlier paper on spectral estimates [Ark. Mat. **1** (1951), 503–531; MR **14**, 187], indicating that the bias and variance of such estimates are mutually antagonistic. He calls these results an uncertainty principle. Lomnicki and Zaremba [J. Roy. Statist. Soc. Ser. B **19** (1957), 13–57; MR **19**, 1098], using a number of heuristic arguments, claim that the principle is invalid. In the opinion of the reviewer, in this paper Grenander effectively disposes of the arguments of Lomnicki and Zaremba. Grenander also refers to analogous results of Whittle [J. R. Statist. Soc. B **19** (1957), 38–47; MR **19**, 1098] and the reviewer [Ann. Math. Stat. **27** (1956), 832–837; MR **18**, 159] paralleling the uncertainty principle.

M. Rosenblatt (Bloomington, Ind.)

2828:

Bass, Jean. Sur certaines classes de fonctions admettant une fonction d'autocorrélation continue. C. R. Acad. Sci. Paris **245** (1957), 1217–1219.

The author considers the class A of complex functions $u(t)$ of the real variable t, $0\leqq t<+\infty$, subjected to the following conditions: (a) $u(t)$ is integrable in every finite interval; (b) $\lim_{T\to\infty} (1/T) \int_0^T u(t)dt=0$; (c) $\gamma(h)=\lim_{T\to\infty} (1/T)\int_0^T u(t+h)\bar{u}(t)dt$ exists and is continuous for all h; (d) $\gamma(0)\neq 0$; (e) $\lim_{h\to\infty} \gamma(h)=0$. From a theorem of H. Weyl [Math. Ann. **77** (1916), 313–352] it is proved that if $\varphi(t)$ is a polynomial of degree $q\geqq 2$, the coefficient of t^q being not commensurable with π, then the function $u(t)=\exp[i\varphi(n)]$ in $n\leqq t<n+1$ (n=non-negative integer) belongs to the class A. *J. Kampé de Fériet* (Lille)

2829:

Bass, Jean; et Bertrandias, Jean-Paul. Moyennes de sommes trigonométriques et fonctions d'autocorrélation. C. R. Acad. Sci. Paris **245** (1957), 2457–2459.

This paper is based on a series of lemmas concerning the function $\gamma(h)$ [see #2808 above], the most important being the following one: if $u(t)=\exp[i\varphi(n)]$ in $n\leqq t<n+1$, then $\gamma(h)$ is continuous and linear in every interval $n\leqq h<n+1$. From these lemmas the authors deduce a demonstration of a theorem, very close to a known result of J. G. van der Corput [Acta Math. **56** (1931), 373–456]: If there exists α, $0<\alpha<1$, and $\theta>0$ such that $|\varphi(n)-\theta n^\alpha|\to 0$ if $n\to\infty$, then

$$\lim_{N\to\infty} (1/N) \sum_{n=0}^{n=N} \exp[i\varphi(n)]=0.$$

J. Kampé de Fériet (Lille)

2830:

Quenouille, M. H. The comparison of correlations in time-series. J. Roy. Statist. Soc. Ser. B **20** (1958), 158–164.

For the (nonparametric) hypothesis that two lengths of time series are generated by the same stationary process, a (parametric) test is developed in which the process is assumed to be autoregressive. An appended remark bows to the spectral approach towards nonparametric tests [see U. Grenander and M. Rosenblatt, Ann. Math. Statist. **24** (1953), 537–558; MR **15**, 448]. *H. Wold* (New York)

NUMERICAL METHODS

See also 2405, 2541, 2642, 3008, 3053.

2831:

✱Goertzel, Gerald; and Kalos, Malvin H. Monte Carlo methods in transport problems. Progress in nuclear energy, Series 1: Physics and Mathematics. Vol. 2, pp. 315–369. Pergamon Press, New York-London-Paris-Los Angeles, 1958. vii+375 pp. **$14.00.**

The target game, as a convenient method of evaluating integrals, and the transport game, useful in the study of certain types of integral equations, are described, with specific application to shielding and reactor core calculations. Preliminary material is given on random and pseudo-random numbers and choice of probability distributions. *I. A. Stegun* (Washington, D.C.)

2832:

Cristescu, Romulus. Method of successive approximations and principal majorants. Gaz. Mat. Fiz. Ser. A. **7** (1957), 337–349. (Romanian)

2833:

Hornecker, Georges. Détermination approchée, à précision numérique élevée, du polynome de meilleure approximation d'ordre n, au sens de Tchebicheff, d'une fonction bornée continue, sur un segment fini. C. R. Acad. Sci. Paris **246** (1958), 43–46.

Let $f(x)$ be defined on a set, S. The best approximating polynomial of degree n for $f(x)$ on S is a polynomial, $\pi_n(x)$, of degree not greater than n such that the maximum residual, $\sup_{x\in S}|f(x)-\pi_n(x)|$, is less than the maximum residual for any other polynomial of degree $\leqq n$. In this brief note, the author proposes a method for obtaining an approximation, $P_n^*(x)$, to the best approximating polynomial of degree n for a function, $f(x)$, continuous on $[-1, 1]$. $P_n^*(x)$ is taken to be the best approximating polynomial for $f(x)$ on the set of $n+2$ points, $x_K^*=x_K+\delta x_K$, where $x_K=\cos[K\pi/(n+1)]$, $K=0, 1, \cdots, n+1$, and $\delta x_K=2(C_{n+2}/C_{n+1})(1-x_K^2)/(n+1)$ if $f(x)$ is neither even nor odd, the C_i being the coefficients in the Tchebycheff expansion of $f(x)$. If $f(x)$ is even or odd, then $\delta x_K=4(C_{n+3}/C_{n+2})(1-x_K^2)X_K/(n+1)$, where n is taken to be one plus the effective degree of the polynomial. The author suggests computing the C_i by means of the relation

$$C_i=2^{1-i}(i!)^{-1}\Big[f^{(i)}(0)+\sum_{v=1}^{\infty}2^{-2v}f^{(i+2v)}(0)/v!(i+1)\cdots(i+v)\Big].$$

As one possible way to calculate $P_n^*(x)$, he suggests the relations $P_n^*(x)=\sum_{i=0}^{n+1}P_i^*L_i^*(x)$, with $1/P_i^*=\omega_i^*\sum_{j=0}^{n+1}(1/\omega_j^*)$, and $\omega_i^*=\prod_{j\neq i}|x_i^*-x_j^*|$, and $L_i^*(x)$, the Lagrange interpolating polynomial for $f(x)$ on the $n+1$ points $x_1^*, \cdots, x_{i-1}^*, x_{i+1}^*, \cdots, x_{n+2}^*$. As an alternative, he offers the solution of the linear system $P_n^*(x_K^*)+(-1)^K l=f(x_K^*)$, $K=0, 1, \cdots, n+1$.

High accuracy is claimed for the method. This claim is not proven, but only illustrated in two simple examples. The assertion to the effect that the author's method is somehow related to the algorithm of E. Remes [same C. R. **199** (1934), 337–340] is not at all clear to the reviewer, unless the relationship is merely that both methods involve the computation of a best approximating polynomial on a set of $n+2$ points. *E. K. Blum* (Los Angeles, Calif.)

2834:

Fiedler, Miroslav. Numerische Lösung algebraischer Gleichungen mit sämtlichen Wurzeln von fast demselben absoluten Betrag. Apl. Mat. **1** (1956), 4–22. (Czech. Russian and German summaries)

This is the case in which some of the classical numerical methods fail to give useful results. In the given equation $f(x)=a_0x^{2s}+a_1x^{2s-1}+\cdots+a_{2s}=0$, let $x=ry$, $y=(a_{2s}/a_0)^{1/2s}$. Then $f(x)=g(y)=b_0y^{2s}+b_1y^{2s-1}+\cdots+b_{2s}$ and there are two polynomials $u(z)$, $v(z)$ in $z=y+y^{-1}$ such that, approximately,

$$y^{-s}g(y)+y^sg(y^{-1})=u(z) \text{ (of degree } s)$$

$$y^{-s}g(y)-y^sg(y^{-1})=(y-y^{-1})v(z).$$

Assuming all roots $z_1, \cdots, z_s$ of $u(z)$ to be simple, let $\varepsilon_i=-v(z_i)/u(z_i)$. Then the roots of $f(x)$ coincide approximately (up to ε^2r, $\varepsilon=\max\varepsilon_i$) with the roots of the quadratic equations

$$x^2-(1+\varepsilon_i)z_irx+(1+2\varepsilon_i)r^2=0.$$

There are a modification of this method for $s=2$ or 3 and four numerical examples.

H. Schwerdtfeger (Montreal, P. Q.)

2835:

Householder, Alston S. A class of methods for inverting matrices. J. Soc. Indust. Appl. Math. **6** (1958), 189–195.

Let P be a nonsingular matrix of order n and let Q be a matrix of rank $n-r$ at most. Then if $P-Q$ has rank $r+1$ at least, there exist column vectors u and v and a scalar σ such that $C=I-\sigma uv^T$ is nonsingular and $C(P-Q)=P-R$, where R has rank $n-r-1$ at most. The author establishes this result and points out that a large class of methods of inverting matrices follows from this result. A criterion for evaluating the effectiveness of various methods is given. *M. R. Hestenes* (Los Angeles, Calif.)

2836:

Pugačev, B. P. On a method of simultaneous calculation of the two limits of a spectrum. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. **1957**, no. 5, 52–70. (Russian)

This paper is concerned with the proof of the theorems quoted as III, V, and VI in the review of an earlier note [Dokl. Akad. Nauk. SSSR (N.S.) **110** (1956), 334–337; MR **18**, 825]. The other theorems announced in that note seem to have been proved in two papers which are unavailable to the reviewer [same Trudy **1957**, no. 3; **1956**, no. 4]. *R. R. Kemp* (Kingston, Ont.)

2837:

Chen, T. C.; and Willoughby, R. A. A note on the computation of eigenvalues and vectors of Hermitean matrices. IBM J. Res. Develop. **2** (1958), 169–170.

W. Given's method for finding the eigenvalues of a real

symmetric matrix (Oak Ridge Nat. Lab. Rep. ORNL-**1574** (1954); MR **16**, 177] is easily adapted to hermitian matrices. The authors point out that, even for hermitian matrices, the last step of Given's algorithm (finding the eigenvalues of a tridiagonal matrix) can be performed in real arithmetic. *P. Henrici* (Los Angeles, Calif.)

2838:

Osborne, Elmer E. On acceleration and matrix deflation processes used with the power method. J. Soc. Indust. Appl. Math. **6** (1958), 279–287.

This paper discusses two programs based on Wilkinson's iterative method [Proc. Cambridge Philos. Soc. **50** (1954), 536–566; MR **16**, 178] for computing the eigenvalues and eigenvectors of complex non-hermitian matrices, as coded for UNIVAC Scientific Computer 1103A. Both of these programs feature floating point, single precision, interpretative complex arithmetic. Both methods utilize matrix deflation. In the first method one forms the sequence $x_{r+1}=k_{r+1}A'x_r$, where $A'=A-\alpha I$, k_{r+1} is chosen to make the component of x_{r+1} of largest absolute value equal to one, and α is a complex parameter chosen to improve convergence. This method handles matrices of order up to 70. A form of the Aitken δ^2-acceleration method is incorporated, but since this is not always satisfactory for complex matrices, a substitute acceleration method is proposed.

The second program results from augmenting the first one with a version of the Wielandt method. Here, $x_{r+1}=k_{r+1}y_{r+1}$, where y_{r+1} is the solution of $(A-\alpha_r I)y_{r+1}=x_r$, and $\alpha_{r+1}=(Ay_{r+1}, y_{r+1})/(y_{r+1}, y_{r+1})$, so that the iterations are carried out with $(A-\alpha_r I)^{-1}$ rather than with $A-\alpha I$. The second method handles matrices of order up to 25 (and after additional programming up to order 75). There is discussion of matrix deflation and the error associated with it. *P. S. Dwyer* (Ann Arbor, Mich.)

2839:

Pope, David A.; and Tompkins, C. Maximizing functions of rotations. Experiments concerning speed of diagonalization of symmetric matrices using Jacobi's method. J. Assoc. Comput. Mach. **4** (1957), 459–466.

The paper is concerned with a discussion of various codifications of the Jacobi method for diagonalizing a symmetric matrix by means of rotations. The novel feature is in the introduction of thresholds which avoid rotations which do not significantly alter the matrix. Four methods are discussed and the convergence of their corresponding iterations is established. The methods were carried out experimentally on a high speed computer and were found to be effective. A description is given of the experimental results obtained.

M. R. Hestenes (Los Angeles, Calif.)

2840:

Gold, Louis. Inverse Bessel functions: Solution for the zeros. J. Math. Phys. **36** (1957), 167–171.

The author proposes to find the roots of $J_n(x)$ by inverse interpolation. The inverse function is approximated by its power series at suitable points, the first few coefficients of which are calculated from the differential equation satisfied by $J_n(x)$.

P. Henrici (Los Angeles, Calif.)

2841:

Orloff, Constantin. Simplification de la méthode de Graeffe au moyen des spectres mathématiques. Bull. Soc. Math. Phys. Serbie **8** (1956), 39–46. (Serbo-Croatian summary)

Le but de ce travail est d'apporter une simplification dans le mécanisme pratique de la méthode de Graeffe pour l'évaluation numérique des racines d'une équation algébrique. La méthode spectrale qui va être exposée simplifie le procédé de transformation d'une équation algébrique en une autre, ayant pour racines les carrés des racines de l'équation donnée, en réduisant le nombre des opérations à effectuer.

La transformation de l'équation

$$\text{(I)} \qquad P(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n=0$$

exige (1) $[\frac{1}{4}(n+2)^2]$ multiplications entre les coefficients, (2) $[\frac{1}{4}n^2]$ multiplications avec le nombre 2, (3) $[\frac{1}{4}n^2]$ additions ou soustractions.

La méthode spectrale réalise la transformation en question, pour n'importe quel n, au moyen: (1) d'une seule multiplication entre les spectres, (2) d'une seule multiplication avec le nombre 2, (3) au plus 3 additions ou soustractions.

La méthode est basée sur le théorème: Soit donnée l'équation (I), a_i étant des nombres entiers. Formons les spectres, ordinaire et corrigé à toutes les places paires, de cette équation, avec le rythme uniforme h, défini par les relations suivantes

$$h=[\log a+\tfrac{1}{2}\log 2(n+1)]+1,\ a=\max|a_i|.$$

Le spectre ordinaire S_1, au rythme uniforme $2h$, de l'équation transformée, aux racines qui sont les carrés des racines de l'équation (I), est égal au produit des spectres

$$S_1=S\cdot\bar{S};\ S=P(10^h),\ \bar{S}=(-1)^nP(-10^h).$$

S. Kulik (Logan, Utah)

2842:

Bridgland, T. F., Jr. A note on numerical integrating operators. J. Soc. Indust. Appl. Math. **6** (1958), 240–256.

Three different finite difference approximations to n-times iterated integrals of a function $f(t)$ are derived by a technique based on systematic use of the "creation operator" E^{-1}, that changes the sequence $(f(0), f(h), f(2h), \cdots)$ into $(0, f(0), f(h), \cdots)$. If $f''(t)$ exists everywhere and if $f(t)\equiv 0$ for $t\leq 0$, the errors of the approximations obtained are $O(h^2)$. The application of these formulas to initial value problems for linear differential equations with constant coefficients leads to methods of approximate solution whose accuracy is $O(h^2)$, where h is the mesh length. As the author points out, his methods have a close relationship to the theory of the so-called z-transformation and, therefore, to that of the Laplace transformation. *W. Wasow* (Madison, Wis.)

2843:

Greenspan, Donald. On a "best" five-point difference analogue of Laplace's equation. J. Franklin Inst. **266** (1958), 39–45.

The author considers 5-point finite difference analogues of Laplace's equation of the form $\sum_{i=0}^4 a_iu_i=0$, where $u_0=u(x, y)$; $u_1=u(x+h, y)$; $u_2=u(x, y+d)$; $u_3=u(x-h,y)$; $u_4=u(x,y-d)$. It is shown that the difference equation $-4u_0+[2/(1+p^2)](u_1+u_3)+[2p^2/(1+p^2)](u_2+u_4)=0$, where $h=pd$, is of third order in the sense that if u is a solution of Laplace's equation, then the difference equation is satisfied except for terms of order higher than third order. The author shows that there does not exist a fourth order difference equation nor an essentially different third order equation such that $\mathrm{Lim}_{d\to 0}\, a_i\neq 0$ for at least one value of i. *D. M. Young, Jr.* (Austin, Tex.)

2844:

Sutyuševa, Š. Š. Estimation of the error in the solution of the Dirichlet problem for the Laplace equation by the method of finite differences. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **21** (1957), 91–95. (Russian)

The author gives a bound for the truncation error in the approximate solution of the Dirichlet problem for Laplace's equation in a plane region, with a sufficiently smooth boundary and sufficiently smooth boundary data, by the simplest difference analog. The bound is based on Gerschgorin's form of the truncation error [Z. Angew. Math. Mech. **10** (1930), 373–382] and on certain estimates for the derivatives of a harmonic function in terms of the derivatives of its boundary values. No proof of these estimates is indicated. *D. G. Aronson* (Flushing, N.Y.)

2845:

Giese, John H. On the truncation error in a numerical solution of the Neumann problem for a rectangle. J. Math. Phys. **37** (1958), 169–177.

Let $U^*(x, y)$ be a harmonic function in a rectangle R such that on the boundary B of R the normal derivative of U^* assumes prescribed values which are in class C^2 on each side of B. If $\int_B (\partial U^*/\partial n)ds=0$ and $U^*=0$ at one corner of B, the solution of this Neumann problem is unique and can be represented as the sum of four Fourier series. The author approximates U^* by the solution u^* of a difference equation problem in a square grid of mesh-length h, based on the familiar five-point formula in the interior and on an approximation of order $O(h^2)$ to the normal derivative at the boundary. The function u^* can be represented by a sum of four finite Fourier interpolation series. By means of a calculation that resembles somewhat an argument for the simpler Dirichlet problem given by the reviewer in the J. Res. Nat. Bur. Standards **48** (1952), 345–348 [MR **14**, 93], it is shown that the truncation error is $O(h^2 \log h)$, uniformly in R.

W. Wasow (Madison, Wis.)

2846:

Bakhvalov, N. S. A method for an approximate solution of Laplace's equation. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 455–458. (Russian)

Let G be a bounded domain in the (x, y)-plane whose boundary Γ consists of a finite number of rectifiable curves. The author considers the problem of solving numerically Laplace's equation $\Delta u=0$ with a Dirichlet boundary condition $u=\varphi$ on Γ. The following unorthodox method is proposed for finding the solution $u^{(h)}$ of the system of the N linear equations arising from the usual finite difference approximation with uniform mesh constant h. The domain G is exhausted by squares of side length $2^k h$ ($k=s, s-1, \cdots$), having their centers at points whose coordinates are integral multiples of $2^{k-1}h$. The components of $u^{(h)}$ corresponding to points inside any such square are expressed in terms of the components corresponding to the circumference of the square by making use of the known analytic expression for the (finite difference) Green's function for the square. Proceeding from bigger to smaller squares, all values of $u^{(h)}$ except those near the boundary can thus be eliminated. There remains a system of $O(N^{\frac{1}{2}} \log N)$ equations which the author proposes to solve by the Jacobi method. The rate of convergence is estimated. The method is said to require less computing time than the (far more general) method of successive over-relaxations due to D. Young [Trans. Amer. Math. Soc. **76** (1954), 92–111; MR **15**, 562], but obviously, the time required for setting up the linear equations is not taken into account in this comparison. {A limited number of copies of a rough translation of the paper are available free of charge from the reviewer.}

P. Henrici (Los Angeles, Calif.)

2847:

Saul'ev, V. K. Numerical integration of parabolic equations. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 36–39. (Russian)

The author illustrates a method for the numerical solution of parabolic equations for the following problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, 0<x<1, t>0; u(x, 0)=f(x); u(0, t)=u(1, t)=0.$$

Also, a modification of the method suitable for the use of high speed computers is given.

S. Kulik (Logan, Utah)

2848:

Hersch, Joseph. Récurrences d'ordre supérieur pour des équations aux différences. C. R. Acad. Sci. Paris **246** (1958), 364–367.

In many characteristic value problems in partial differential equations, the characteristic function is often determined, by finite-difference methods, with greater accuracy than the characteristic value, and in some cases the function, but not the value, is identical with the solution of the differential equation. In the latter case the characteristic value can also be determined exactly, and in any case more accurately, by a new "deferred approach to the limit" method, based on calculated results at two different intervals. Examples are given for membranes of various shapes, and for square and triangular meshes. *L. Fox* (Teddington)

2849:

Velasco de Pando, Manuel. The method of regula falsi for solving integral equations. Rev. Acad. Ci. Madrid **51** (1957), 139–147. (Spanish)

In order to solve Fredholm integral equations $y=\lambda Ky+f$, the author proposes and discusses various methods, all of which are related to a complete set of functions $m_1, m_2, \cdots$. Linear combinations of these functions are used either to approximate or to represent the solution y when λ is not an eigenvalue of K. The first method uses linear combinations $\mu_1, \mu_2, \cdots$, such that $\tau_i=\mu_i-\lambda K\mu_i$ and $(\tau_i, \tau_k)=\delta_{ik}$. An expansion $f=\sum c_i\tau_i$ implies $y=\sum c_i\mu_i$. It may be mentioned that this method was originally proposed by D. Enskog more than 40 years ago.

In the second method the author sets $t_i=m_i-\lambda Km_i$ and minimizes the Hilbert norm of $f-\sum c_i t_i$ by variation of the coefficients c_i. The minimizing coefficients lead to $y=\sum c_i m_i$. Finally, the author considers the set $m_i=Km_{i-1}$ and makes some remarks about the related Neumann-Liouville iteration series.

H. F. Bückner (Madison, Wis.)

2850:

✱Giet, A. Abacs or nomograms. An introduction to their theory and construction illustrated by examples from engineering and physics. Translated and revised by J. W. Head and H. D. Phippen. Iliffe & Sons Ltd., London; Philosophical Library, New York; 1956. ix+225 pp. $12.00.

Cet ouvrage est la traduction d'un ouvrage français paru en 1954 [Dunod, Paris; MR **17**, 91].

J. Kuntzman (Grenoble)

2851:

Pentkovskiĭ, M. V. On the estimation of the error in calculation by nomograms from aligned points and on their best transformation. Akad. Nauk Kazah. SSR. Trudy Sektor. Mat. Meh. 1 (1958), 62–70. (Russian)

The author divides the sources of errors in using nomograms for the equation $F(u, v, w)=0$ with three scales with aligned points in four classes: 1: those originating from the interpolation on the scales for u and v; 2: those occurring by adjusting the solving straight line; 3: those from the intersection of the solving straight line with the scale of w; 4: those from the interpolation on the scale of w. Each of these errors, and the total error, is estimated. The "best nomogram" is defined to be that which does not permit the total error to exceed a predetermined fixed upper bound in the smallest geometrical dimensions. Besides these, the author introduces the "admissible nomograms" as those for which a predetermined fixed error shall not be exceeded within a given geometrical domain, this domain not being the smallest possible. The general problem is discussed in general terms. Only the projective transformations in nomograms with three parallel scales is studied in detail.

E. M. Bruins (Amsterdam)

2852:

✶Montagne, P. Tables abrégées de puissances entières, spécialement préparées pour servir d'aide à la machine à calculer. Dunod, Paris, 1958. xliv+411pp.+32pp. (tables annexes). 5600 francs.

This is a somewhat monumental table of integral powers x^n; it is considerably the most extensive published. It has been reproduced photographically from manuscript — the calligraphy is remarkable, though it remains less easy to read than printing from type.

The tables are divided into three main sections.

	x	n
Petites tables:		
15 figures.	0.2	1(1)600
	0.3	1(1)400
	0.4	1(1)250
	0.5(0.1)0.9	1(1)200
	1.1(0.1)2	1(1)150
	0.1(0.01)1.53	1(1)78
Tables moyennes:		
15 figures.	0.1(0.001)1.26	1(1)26
Grandes tables:		
10 figures.	0(0.0001)1	1(1)10
8 figures.	1(0.0001)1.26	1(1)10

In all tables the final figure is followed by a remarkable variety of odd symbols giving information about the subsequent digits — a single extra digit would have been more informative and much more elegant.

The introduction discusses arrangement, construction and use of the tables.

There is also a separate booklet of tables annexes, which includes material and tables which it is convenient to use detached from the main book. This includes a table of binomial coefficients $\binom{n}{r}$ for $r=1(1)10$, $n=1(1)150$; a table of Everett interpolation coefficients for fractions 0(.002)1 of the tabular interval — coefficients E_2, F_2, E_4, F_4 with differences; and another of Everett coefficients for fractions 0(.01)1 for use with differences up to δ^{14}. There are also 20-figure tables of the first 100 factorials and their reciprocals; a 6-decimal table of a/b, for a, $b=1(1)26$ independently, arranged in order of magnitude; a 13-figure table of $10^{a/b}$, $b=1(1)26$, $a=1(1)b$; and a number of 20-figure values of multiples and powers of special constants: π, e, etc.

J. C. P. Miller (Cambridge, England)

2853:

✶Rogers, William M.; and Powell, Robert L. Tables of transport integrals

$$J_n(x)\equiv\int_0^x \frac{e^z z^n dz}{(e^z-1)^2}.$$

Circular 595, National Bureau of Standards, Washington, 1958. ii+46 pp. $0.40.

These tables give 6-figure values of the integrals for $n=2(1)17$, $x=.1(.1)X$, where X is such that $J_n(X)=J_n(\infty)$ to 6 figures. X varies from 25 at $n=2$ to 40 at $n=17$. Also given are 6-figure values of $J_n(x)/x^{n-1}$ for $n=2(1)7$ and the same values of x. No differences are given. There are also 8-figure values of $J_n(\infty)=n!\zeta(n)$, for $n=2(1)17$, where $\zeta(n)$ is the Riemann zeta-function.

An introduction gives a derivation of three series used in the computations and discusses accuracy and interpolation. *J. C. P. Miller* (Cambridge, England)

2854:

✶Tables of the exponential integral for complex arguments. National Bureau of Standards Applied Mathematics Series, No. 51. U. S. Government Printing Office, Washington, D.C., 1958. xiv+634 pp. $4.50.

This massive and long-awaited book of tables can be simply described. Table I gives 6-decimal values of

$$E_1(z)=\int_z^\infty \frac{e^{-u}}{u}\,du$$

for $z=x+iy$, $x=0(.02)4$, $y=0(.02)3(.05)10$. Tables II and III give 6-decimal values respectively of $E_1(z)+\ln z$, $x=0(.02)1$, $y=0(.02)1$, and of $e^zE_1(z)$, $x=4(.1)10$, $y=0(.05)10$, while Table IV gives 10-decimal values of $E_1(z)$, $-x=0(.1)3.1$, $y=0(.1)3.1$, and of $E_1(z)+\ln z$, $-x=0(.1)1$, $y=0(.1)1$. Finally, Table V gives 6-decimal values of $e^zE_1(z)$ for $x=-20(1)20$, $y=0(1)20$, and $-x=0(.5)10$, $y=0(.1$ or $.5)4(.5)10$. An introduction discusses definitions and the calculation of and interpolation in the tables, and gives a bibliography of a few related tables. *J. C. P. Miller* (Cambridge, England)

2855:

✶Boll, R. H.; Leacock, J. A.; Clark, G. C.; and Churchill, S. W. Tables of light-scattering functions: relative indices of less than unity, and infinity. University of Michigan Press, published for The Engineering Research Institute, Ann Arbor, 1958. viii+360 pp. $9.50.

This book gives 6-decimal tables of the real and imaginary parts of the functions

$$A_n=\frac{(-1)^{n+\frac{1}{2}}(2n+1)}{n(n+1)}\,\frac{S_n(\alpha)S_n'(m^*\alpha)-m^*S_n(m^*\alpha)S_n'(\alpha)}{\phi_n(\alpha)S_n'(m^*\alpha)-m^*S_n(m^*\alpha)\phi_n'(\alpha)},$$

$$B_n=\frac{(-1)^{n+1\frac{1}{2}}(2n+1)}{n(n+1)}\,\frac{m^*S_n(\alpha)S_n'(m^*\alpha)-S_n(m^*\alpha)S_n'(\alpha)}{m^*\phi_n(\alpha)S_n'(m^*\alpha)-S_n(m^*\alpha)\phi_n'(\alpha)},$$

in which $S_n(\alpha)$, $\sqrt{(\frac{1}{2}\pi\alpha)}J_{n+\frac{1}{2}}(\alpha)$, and

$$\phi(\alpha)=\sqrt{(\tfrac{1}{2}\pi\alpha)}\{J_{n+\frac{1}{2}}(\alpha)+i(-1)^nJ_{-n-\frac{1}{2}}(\alpha)\}$$

are Riccati-Bessel and Riccati-Hankel Functions and $m^*=m-ik$; the tables appear to take $k=0$.

Values are given for $m=0.6, 0.7, 0.75, 0.8, 0.9$,

$$\alpha=1(1)10(2)20(5)100(10)160(20)200;$$

and for

$$m=0.93,\ \alpha=1(1)5(5)80,\ 95,\ 115,\ 135,\ 160,\ 200;$$

in each case $n=1(1)N$, where A_N, B_N are zero to six decimals; $N=5$ at $\alpha=1$, up to $N=217$ at $\alpha=200$.

Also given are 5-figure values of K for $m=0.6, 0.7, 0.75, 0.8, 0.9$; and 4-figure values for $m=0.93$, for each corresponding value of α used above, where

$$K=\frac{2}{\alpha^2}\sum_{n=1}^{\infty}\frac{n^2(n+1)^2}{2n+1}(|A_n|^2+|B_n|^2).$$

These functions are used in the light-scattering theory of Mie, and there have been a number of recent tables connected with this theory — several are listed in the list of references in the book.

J. C. P. Miller (Cambridge, England)

2856:

✱Subtabulation: a companion booklet to Interpolation and allied tables. Her Majesty's Stationery Office, London, 1958. 54 pp. (1 insert) 7s. 6d.

This booklet is a companion booklet to "Interpolation and Allied Tables", which originated as a reprint, by L. J. Comrie, from the Nautical Almanac for 1937. It provides methods and tables, suitable for use with desk computers or in hand calculations, for obtaining from a table of a function, given at a relatively wide argument interval, a more extended table of the same function at a smaller argument interval.

The booklet has four parts. Part I is an introduction giving a comparison of methods, notation and a discussion on accuracy, together with a bibliographical note.

Part II deals with direct methods. These are undoubtedly the simplest to use and to understand. Tables are provided for interpolation to 20ths and to 24ths, using various formulae: Bessel's formula to δ^4 (with $\delta^4<500$); the modified Everett-Bessel formula to terms in modified fourth differences (to a limit of about 10^6 in δ^4); Lagrange's formula with 6 points (8-decimal coefficients) and 10 points (10 decimal coefficients). More extended formulae for interpolation to halves, thirds and tenths are considered, and "throw back" factors are given for preparing coefficients in various modified formulae.

Part III gives a new method of Precalculated Second Differences, with the necessary tables for subtabulation to fourths, sixths and tenths.

Part IV outlines the method of Bridging Differences, so effective when certain accounting machines, such as the National Accounting Machine, models 3000 or 37, are available.

There is also an interesting Index, listing also the formulae and coefficients referred to.

The three methods outlined, Direct, Precalculated Second Differences, and the Method of Bridging Differences, are in order of increasing efficiency, and also of increasing complexity, the last requiring rather special equipment for full efficiency. It seems a pity that the End-Figure Method, introduced by Hudson in the Nautical Almanac Office and developed by Comrie (see the Nautical Almanac for 1931) into a simple and most effective tool, was not also included. It is simpler to learn than the method of Precalculated Second Differences, which is said in the booklet to supersede it, and is quicker in use in at least some applications, though more restricted in its application.

This is a useful set of tables for those not having access to an automatic computer.

J. C. P. Miller (Cambridge, England)

COMPUTING MACHINES

See also 2838, 3041.

2857:

Nadler, Morton. Introduction to digital computers. A survey of computer types. Apl. Mat. 2 (1957), 409–423. (Czech. Russian and English summaries)

General review of aids to computation, analogue and digital. Various types of computers are briefly described. There are four types of computing laboratories: those concerned with design of computers and their components (IBM, NBS, Institute of Numerical Technique at the Academy Nauk SSSR), research institutes using commercial equipment (Oak Ridge, Institute for Nuclear Physics SSSR), computing centres attached to institutes for applied mathematics (Computing Centre of Academy Nauk SSSR) and industrial technical centres (Zeiss, Jena).

There are 29 references, 6 to Russian papers.

V. Vand (University Park, Pa.)

2858:

Kitz, Norman. Possible future trends in computing machines. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **423** (1955), 6 pp.

2859:

Burks, Arthur W.; and Wang, Hao. The logic of automata. I, II. J. Assoc. Comput. Mach. **4** (1957), 193–218, 279–297.

The authors are interested in the "structure and behavior of automata". To this end they characterize them in terms of various kinds of state tables providing decision procedures. The paper consists of two major portions. In the first part the emphasis is on automata and nets. These latter are, in effect, a diagrammatic device for exhibiting the internal structure of an automaton. It is shown how one can proceed from nets to state tables and conversely. In part II the notion of a transition matrix is introduced and analyzed. This notion arises in the analysis of a net and, in particular, in transitions from one state to another.

H. H. Goldstine (Princeton, N.J.)

2860:

Svoboda, Antonín; and Valach, Miroslav. Operational circuits. Stroje na Zpracování Informací 3 (1955), 247–295 (1956). (Czech. Russian and English summaries)

The basic theory of generalised calculus of digital computing circuits is discussed. In the design of hardware for multiplication, the product is usually formed as a sum of multiples of the multiplicand. Valach [Stroje na Zpracování Informací 3 (1955), 211–245; MR 19, 1085] has shown that circuits are possible which add and multiply in a single cycle. This work shows that not only multiplication, but also higher operations can be performed in one working cycle of a circuit by comparatively simple means, if they are represented in the physical world of hardware by algorithms derived from number theory.

Number systems are discussed. The number system of remainder classes represents whole numbers mod P, where P is a product of definitely chosen numbers. Transformation of number representations from the system of remainder classes to some polyadic system is discussed. Circuits are much simplified if a concept of an "estimate" is introduced to aid the transformation. The representation of numbers in the system of remainder classes and the representation of operations by the use of operational circuits are described. Relay circuits for

the transformation of numbers by the method of "estimates" are described. *V. Vand* (University Park, Pa.)

2861:

Svoboda, Antonín. Single impulse time dyadic relay adders. Stroje na Zpracování Informací 3 (1955), 297–308. (Czech. Russian and English summaries)

A purely theoretical paper, in which the basic concepts and theory of binary adding circuits (computer hardware) for single-cycle operation are analysed. Circuits for single-cycle addition of more than two numbers are so comparatively complex and, therefore, expensive that they have not yet found practical application. It is, however, of theoretical interest to discuss the design of such circuits. A single-cycle circuit for the addition of four six-digit numbers is shown in detail.

V. Vand (University Park, Pa.)

2862:

Línek, Allan; and Novák, Ctirad. Computing machines of the Laboratory for Crystal Structure, Institute of Technical Physics, Czechosl. Academy of Science. Stroje na Zpracování Informací 3 (1955), 309–321 (1956). (Czech. Russian and English summaries)

Four special-purpose computing machines are under construction: a machine for the first stage of crystal structure computation, which computes structure factors from atomic positions and compares them with the observed structure factors; two machines for Fourier synthesis of electron density at points not further apart than 1/120 of the cell edge; and a machine for computation of the trigonometrical values of cos $2\pi(hx+ky+lz)$ and corresponding sin. All of the machines are of the relay type and work in the binary system.

V. Vand (University Park, Pa.)

2863:

Dady, Guy. Le calcul électronique apporte une optique nouvelle sur un problème ancien: La prévision météorologique. Chiffres 1 (1958), 49–61.

This article describes a computer ordered by the French Meteorological Service for use in numerical weather forecasting.

After reviewing the traditional subjective method of forecasting and the procedures used more recently in numerical weather forecasting, a survey of computer requirements for the weather forecasting problem is given. A large memory is needed because the volume of input is very large, and also because the usual method of solution of the numerical problem requires extensive calculations. Usually this involves solving the difference analogue of a system of Poisson or Helmholtz differential equations by the Leibmann method of relaxation.

Input facilities which would handle this information without manual translation would be highly desirable. In order to achieve this, the International Meteorological Code should be revised to a form amenable to direct computer input and the computer should be able to read this source document.

The French Meteorological Service has ordered a calculator which they feel meets many of their requirements, and which has the following characteristics: 1) It is equipped with a rapid access memory composed of six groups, each of 1,024 words. Each word contains 29 binary digits which are manipulated in parallel. 2) It has a slow speed memory consisting of three magnetic tapes which could be increased to eight. 3) It has an arithmetic organ which operates in floating or fixed point mode. In floating point mode, it handles numbers in magnitude less than one and greater than 10^{-14}. 4) Its input devices include keyboard, photo-electric perforated tape reader and magnetic tape reader. 5) Output devices include a paper tape punch and magnetic tape writer. 6) 29 input/output terminals allow external access to the arithmetic unit.

The machine cycle is five microseconds; some overlapping of operations is allowed by virtue of independent controls. The author believes that the impact of electronic computers on meteorology will be dependent on the psychological reaction of meteorologists to the replacing of their special talents with objective analyses, and the speed with which the International Meteorological Code can be changed. *J. F. Blackburn* (New York, N.Y.)

2864:

✱Gotlieb, C. C.; and Hume, J. N. P. High-speed data processing. McGraw-Hill Series in Information Processing and Computers. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. xi+338 pp. $9.50.

This book is not a mathematics book, but it could be useful to a mathematician interested in learning how far one off-shoot of mathematics has moved from its fundamentalorigins. The authors have covered, in a thorough fashion, applications of digital computers — high-speed data processors — to generally non-scientific, business oriented problems. The book and its authors are not mathematically naive, and their presentation gains from its relationship to a formalized mathematical background. Some of its useful points are a detailed description of a real, but relatively simple computer, discussion of the use of differences, automatic checking procedures, and some simple optimization of coding techniques. A thorough discussion of sorting will be helpful in many practical business problems. It still leaves completely intuitive what the relationship between user and machine really is, and this process of communication is only described by example. A mathematically oriented book on the algorithms of the present stored program calculators is apparently as yet unwritten. *J. W. Carr III* (Ann Arbor, Mich.)

2865:

Brooker, R. A. Further autocode facilities for the Manchester (Mercury) computer. Comput. J. 1 (1958), 124–127.

The more advanced facilities described here [see Brooker, same J. 1 (1958), 15–21 for original description of the system] include the following: (1) the choice of unrounded rather than rounded arithmetic; (2) an instruction for selecting the maximum or minimum of a set of elements; (3) a method for the step-by-step integration of a system of ordinary differential equations using the Runge-Kutta procedure; (4) the technique for building up long programs consisting of several 'chapters' including the method of redefining variables; (5) a method of supplementing the initial group of 495 variables by introducing auxiliary variables as required; (6) a method of making efficient use of short routines; e.g., 'subchapters' (part of a 'chapter' used as a closed subroutine) and 'quickies' (routines for evaluating standard functions permanently available on the drum); (7) a system for operating with complex numbers.

C. C. Gotlieb (Toronto, Ont.)

2866:

Fotheringham, J. A.; and Roberts, M. de V. An input routine for the Ferranti Mercury computer. Comput. J. 1 (1958), 128–131.

The Mercury computer is described briefly (it is a binary machine with both core and drum stores), and the assembly system intended for those who use machine language for programming is then described. The scheme simplifies coding and adopts certain conventions, but since it reflects the construction of the machine, it imposes a minimum of restrictions. The input system retains the general method used on the Mk I computer of bringing, in turn, segments of a program from the drum onto the high speed store, allowing space for working area, and providing a routine changing sequence for carrying out the transactions between segments. Among the features of the input system are: (1) Simplification of the notation for instructions by adopting a mnemonic pair of decimal digits for the operations and a convenient system for the addresses. (2) The use of symbolic address. (3) A natural provision for routines and subroutines by dividing programs into "chapters", a group of routines held simultaneously in the high speed store; "sub-chapters", part of chapters used as closed subroutines; and "quickies", frequently used subroutines permanently available on the drum. (4) Directives for controlling the action while reading tape. In addition to chapter, routine and quicky these include "up" and "down" for storing and writing, "enter" for initiating a program, and some others. (5) Facilities for entering constants and parameters. (6) Methods for printing titles, inserting corrections, printing out the addresses of routines and chapters, and introducing special features. *C. C. Gotlieb* (Toronto, Ont.)

2867:

Böhm, Corrado. Sulla programmazione mediante formule. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **423** (1955), 9 pp.

Brief description of a problem-oriented language.
A. S. Householder (Oak Ridge, Tenn.)

2868:

Vitushkin, A. G. Some estimates from the tabulation theory. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 923–926. (Russian)

2869:

Marczyński, Romuald. Generator with stabilised power output for network analysers. Stroje na Zpracování Informací **3** (1955), 323–327 (1956). (Czech. Russian and English summaries)

A constant power generator for *AC* network analysers is described. It is suitable both for steady state and transient studies. *V. Vand* (University Park, Pa.)

2870:

Šafránek, Milan. Czechoslovak network analysers. Stroje na Zpracování Informací **3** (1955), 329–370 (1956). (Czech. Russian and English summaries)

Two analogue network analysers, developed in Czechoslovakia, are described. The universal *DC* analyser, which was completed in 1952, has 130 model units composed of decade resistors. A large *AC* model is being completed. Its supply is rated 500 cycles, 100 Volt, 100 mA. There are 24 generators, 144 transmission lines, 48 unit loads, 24 capacitive units, 12 autotransformers and 12 transformers. *V. Vand* (University Park, Pa.)

2871:

Aoyagi, Kenji; and Miyawaki, Kazuo. Electric Fourier transformer. Tech. Rep. Osaka Univ. **7** (1957), 31–36.

The authors have constructed an automatic special purpose analog-digital computer for computation of the Fourier transformation

$$W(f)=\int_0^\infty \varphi(\tau)\cos 2\pi f\tau d\tau.$$

The curve to be transformed is plotted on a grating mask and placed in front of a cathode ray tube. A vertical beam intersects the grating mask and gives out a series of digital pulses which are read by a photo-multiplier tube. The transforming function (which may, for example, be a Bessel function, rather than a harmonic function) is stored in the form of digital pulses on a magnetic tape. Digital integration by means of a dekatron counter follows a simple relay multiplication. Two examples, with actual and machine plots of transforms, are shown.
J. W. Carr III (Chapel Hill, N.C.)

MECHANICS OF PARTICLES AND SYSTEMS

See also 2494, 2977.

2872:

Blaschke, Wilhelm. Zur Kinematik. Abh. Math. Sem. Univ. Hamburg **22** (1958), 171–175.

Let $K(t)$ be an instantaneous position of a rigid body in stationary Euclidean space K', and $S(t)$ the axis of an infinitesimal screw which displaces $K(t)$ into $K(t+dt)$. Using the canonical coordinate systems $\mathbf{k}(t)$, $\mathbf{p}_j(t)$ in K (with origin $\mathbf{k}$ and the unit vectors $\mathbf{p}_j$, $j=1, 2, 3$, where $\mathbf{p}_1$ lies on the $S(t)$ and $\mathbf{p}_3$ is orthogonal on $\mathbf{p}_1$ and $\mathbf{p}_1+d\mathbf{p}_1$) $\mathbf{k}'(t)$, $\mathbf{p}_j'(t)$ in K', author derives "the derivative equations" with "guide conditions" (for one point in K) and "conditions of rest" (for one stationary point in K'). Introducing a canonical parameter (s), the above equations are expressed by means of this parameter, and the relations for $\dot{x}$, $\ddot{x}$ and $\dddot{x}$ are derived, where $\dot{x}=dx/ds$. Further, it is shown that for points in $K(t)$ whose trajectories have stationary osculatory planes, the determinant $[\dot{x}, \ddot{x}, \dddot{x}]=0$ vanishes, giving one surface of the 3rd order. The motion is of cylindric kind. The vector $\mathbf{p}_1$ rests in space K and the components of $\dot{x}$ and $\ddot{x}$ show that the motion on a plane $x_1=0$ is a classical slide rod motion. The author concludes that, by means of derived equations, this and other properties of G. Darboux's cylindric motion [C. R. Acad. Sci. Paris **92** (1881), 118–121] can be proved. *D. Rašković* (Belgrade)

2873:

Rumyancev, V. V. Rotational stability of a rigid body with a fluid-filled ellipsoidal cavity. Prikl. Mat. Meh. **21** (1957), 740–748. (Russian)

The author claims [Trudy Inst. Meh. Akad. Nauk SSSR (1956), no. 2] to have given sufficient conditions for the stability of rotation about the vertical of a heavy rigid body of revolution, having an ellipsoidal cavity, filled completely with a perfect fluid in a state of homogeneous vortex motion.

Using the so-called Lyapunov V-functions, in the present paper the author investigates the stability of the unperturbed motion of a rigid body with a three-axial ellipsoidal cavity, filled with a fluid as above. Also, certain particular cases are discussed. For the details of the results the reader is referred to the paper itself.
E. Leimanis (Vancouver, B.C.)

2874:

Bottema, O. On Staude's motion in five dimensional-space. Nederl. Akad. Wetensch. Proc. Ser. A. **60**= Indag. Math. **19** (1957), 248–253.

Staude's theorem concerns the motion in ordinary three dimensional Euclidean space of a rigid body about a fixed point under the influence of gravity. It states that uniform rotation about certain axes is always possible. The present paper gives analogous results for five dimensional Euclidean space. Unlike the situation in the three dimensional case, there are two different types of uniform rotation. *D. C. Lewis, Jr.* (Baltimore, Md.)

2875:

Tatarkiewicz, Krzysztof. Une généralisation des équations de Maggi et d'Appell. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **10** (1956), 5–32 (1958). (Polish and Russian summaries)

The motion of a dynamical system subject to constraints depending on parameters defining position and its derivatives up to an order higher than two, and on time are analyzed. For the "not-working constraints", i.e., for the class of constraints which are usually called "ideal", the Lagrange equations of the 1st and 2nd kind are derived, as well as Appell's equations.

But it must be remarked that the same problem is discussed and almost the same results are obtained for an even more general class of constraints by R. Kašanin [Publ. Inst. Math. **2** (1948), 116–130; MR **10**, 489]. There are certain differences, but they are not essential.
T. P. Anđelić (Belgrade)

2876:

✶**Minorsky, N. The theory of oscillations. Dynamics and nonlinear mechanics.** Surveys in Applied Mathematics, Vol. 2, pp. 109–197. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. xii+206 pp. $7.75.

This survey is a quite brief and elementary account of some recent developments in the theory of oscillations. The survey is not intended for specialists, nor is it a source for precisely stated results. The survey gives a general idea of some of the principle problems and methods. The author, who has already contributed so much to this field and has done so much to call attention to the work of Soviet mathematicians, has attempted to reach as broad an audience as possible, and this article should prove to be a valuable introduction to the theory of nonlinear oscillations, particularly for those interested in applications. The survey also calls attention to methods which have been highly developed in the USSR, but relatively neglected in the West. A particularly good example of this is the author's remarks on Liapounoff's second method. The author also makes a point of indicating a number of places where the mathematical theory lags far behind problems raised by the study of actual physical systems. In spite of this, there have been great advances in the theory of differential equations which should have many important applications, and it is hoped that this article will stimulate some engineers to inform themselves better on more recent developments. A translation of a number of the fine new books published in the USSR and the writing of some of our own is badly needed.
J. P. LaSalle (Baltimore, Md.)

2877:

✶**Leimanis, E. Some recent advances in the dynamics of rigid bodies and celestial mechanics. Dynamics and nonlinear mechanics.** Surveys in Applied Mathematics, Vol. 2, pp. 1–108. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. xii+206 pp. $7.75.

This article, together with Minorsky's on nonlinear mechanics (see preceding review), is Volume II of the Surveys in Applied Mathematics being written as a joint project of the Office Of Naval Research and Applied Mechanics Reviews. As described in the preface by F. Joachim Weyl, the survey articles have the primary objective of summarizing the state of selected fields of applied mathematics, particularly those fields whose current state cannot be satisfactorily assessed without a study of the less accessible literature on the subject. The articles are not written for specialists in the field, but are aimed at a fairly broad audience, including engineers.

Leimanis' survey of dynamics is well organized and, especially when one considers the limitations of a survey, his presentation of the topics he selected is both interesting and complete. He does particularly well in giving an account of some of the recent Soviet and Italian contributions. The chapter headings and the principle topics covered are: Motion of a rigid body about a fixed point (the Euler and Poisson equations; the Hess and Schiff equations; the Euler space; manifold of configurations and time); mathematical exterior ballistics (integrability and integration of the equations of motion of a projectile considered as a particle; equations of motion of a projectile considered as a rigid body); uniformization of the three-body problem; capture in the three-body problem (Chazy's, Shmidt's, Hil'mi's, and Merman's investigations); generalized n-body and three-body problems. There is a carefully prepared bibliography at the end of each chapter. *J. P. LaSalle* (Baltimore, Md.)

2878:

Akashi, Hajima; and Levy, Sheldon. The motion of an electric bell. Amer. Math. Monthly **65** (1958), 255–258.

2879:

Gulyaev, M. P. On dynamically possible, regular precessions of a solid body with one fixed point. Akad. Nauk Kazah. SSR. Trudy Sektor. Mat. Meh. **1** (1958), 202–208. (Russian)

The author points out that the regular precession, under the action of gravity, of a solid body with one fixed point has been treated extensively in the literature. There remain, however, certain conditions that have not received sufficient attention, but under which motion is possible. The present work is devoted to the consideration of these conditions and to a proof of the uniqueness of the known regular precessions.
H. P. Thielman (Ames, Iowa)

2880:

Schieldrop, Edgar B. Démonstration mécanique d'une propriété de la précession du gyroscope. Bull. Sci. Math. (2) **81** (1957), 171–174.

2881:

Fedorčenko, A. M. On the motion of an asymmetric heavy gyroscope with a vibrating fulcrum. Ukrain. Mat. Ž. **10** (1958), no. 2, 209–218. (Russian. English summary)

In a previous paper [same Ž. **9** (1957), 220–224; MR **19**,

785] the author described a special canonical method of averaging for a mechanical system whose Hamiltonian depended on time. In the present paper that method is shown to be applicable to the case when the Hamiltonian of the system depends on a certain angular variable ϕ. The problem of the motion of a rapidly rotating asymmetric heavy gyroscope with a vibrating fulcrum is then considered by this method. The following qualitative features of the motion of the mechanical system investigated are obtained.

1. The averaged motion of an asymmetric heavy gyroscope with a vibrating fulcrum is the same as that of a symmetric heavy gyroscope, except for a difference in the regions of validity.

2. Because of the nonlinearity of the system, and due to the periodicity of the Hamiltonian with respect to ϕ, the external periodic forces may give rise to resonance phenomena if the frequency of the external force satisfies certain conditions. The resonance will vanish if the center of gravity of the gyroscope is located on the axis of inertia. *H. P. Thielman* (Ames, Iowa)

2882:

Bishop, R. E. D.; and Johnson, D. C. On damped free vibration with particular reference to systems having nearly-equal natural frequencies. Aero. Quart. **9** (1958), 71–95.

This paper concerns the free vibration of linear systems subjected to damping. From a mathematical point of view the paper is purely expository as it contains no non-trivial result which is not well known.

D. C. Lewis, Jr. (Baltimore, Md.)

2883:

***Egorov, V. A. Certain problems of moon flight dynamics.** The Russian literature of satellites. I, pp. 107–174. Translated from Uspehi Fiz. Nauk **63** (1957), no. 1a. International Physical Index, Inc., New York, 1958. vi+181 pp. (1 plate) $10.00.

The paper under review presents the basic results of work on a number of important problems in moon flight theory carried out during the years 1953–1955 at the Institute for Mathematics of the Academy of Sciences of the USSR. A brief communication of some of these results was published in a recent paper by the author [Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 46–49; MR **19**, 994].

The first eight pages give a list of symbols and definitions which are used in the paper. Due to the complexity of notations used and results obtained, it is almost impossible to give a detailed technical review of the paper. However, a brief non-technical description of the section contents will give an indication as to the problems investigated.

In Section 1 the problem of moon flight is reduced to the restricted circular three-body problem of the earth, the moon and the rocket. (Throughout the paper the term "bounded circular three-body problem" is used, obviously as a result of an incorrect translation from the Russian.) In Section 2, following Hill's method, the Jacobi integral of the equations of motion is used to determine the minimum velocities required for reaching the moon and for capture of an earth projectile by the moon.

Section 3 is concerned with the actual evaluation of the minimum velocity trajectories by the method of numerical integration.

Sections 4, 5 and 6 examine the possibility of the moon capturing a projectile launched from the earth, give an approximate method of studying approach trajectories (i.e., those trajectories which begin near the earth and enter the sphere of action of the moon on their first revolution around the earth), and analyze the evolution of the complete set of approach trajectories when the magnitude and direction of the initial velocity is varied. It is shown, for example, that it is impossible to obtain a permanent moon satellite on the first revolution without any additional thrust.

In Sections 7, 8 and 9, methods are developed for solving the problems of hitting the moon, circumnavigating it and circumnavigating it with a flat re-entry into the earth's atmosphere. Also, a method is developed for determining the initial conditions for a return to the earth, and a classification is given of circumnavigational trajectories and approach trajectories which return to the earth but do not circumnavigate the moon.

In Section 10 the problem of periodic circumnavigation of the moon is examined. Finally, in Section 11 the problem of producing the maximum acceleration of a cosmic rocket by means of the moon's pull without making use of the thrust of the motor is discussed.

At the end of the paper, possible ways are indicated for generalizing the methods developed and the results obtained, with particular reference to the problem of flight from the earth to the outer planets of the solar system. *E. Leimanis* (Vancouver, B.C.)

2884:

Carrara, Nello; Checcacci, Pier F.; e Ronchi, Laura. Sulla determinazione della traiettoria dei satelliti artificiali. Ricerca Sci. **28** (1958), 1341–1355.

Si descrive un metodo basato sull'effetto Doppler per la determinazione di alcuni parametri della traiettoria di un satellite artificiale; si discute la precisione raggiungibile con tale metodo, quindi si descrive un sistema per la determinazione della traietoria, che presenta il vantaggio di non aver bisogno di nessuna ipotesi preliminare sul moto in esame. *Riassunto dell'autore*

2885:

Marsicano, F. R. On the equations of motion of the satellite. Boll. Un. Mat. Ital. (3) **13** (1958), 214–216. (Spanish)

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 2969, 3027.

2886:

***Kittel, C. Elementary statistical physics.** John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London; 1958. x+228 pp. $8.00.

Within the small framework of this book, the author sets himself an ambitious task — nothing less than to cover at least the elements of thermodynamics and statistical mechanics, both for equilibrium states and irreversible processes. This much material could easily fill four texts, each of them larger than the present volume; nevertheless, especially in the first half of the book, one does not get an impression of excessive condensation. The foundations of statistical mechanics are lucidly explained, and this is followed by an excellent discussion of classical thermodynamics which almost, but not quite, manages to avoid those two concepts which

cause so much trouble to the beginning thermodynamics student, heat and work. Each introduction of a new idea is followed by one or more examples worked out in detail, plus a number of exercises. The canonical and grand canonical ensembles are then discussed, again with numerous applications, and the section on equilibrium properties is brought to a close with a brief presentation of the density matrix.

The following part of the book is on fluctuations, and includes a somewhat disproportionate number of pages devoted to the response of electrical circuits to random inputs. Other topics discussed here include the Fokker-Planck equation and the Onsager relations.

The third and last section is on kinetic theory, and starts with a presentation of the H-theorem, followed by the principle of detailed balance, with applications to nuclear reactions, photoionization, and semiconductors. The other major topic here is the Boltzmann equation for transport processes, in which the collision term is unfortunately replaced by a simple relaxation; this introduces a relaxation time parameter which could have been calculated from the complete Boltzmann theory. Finally, there are appendices on the method of steepest descents, the Dirichlet discontinuous factor, computer methods in molecular dynamics, and the virial theorem.

On the whole, this text is an excellent one for a short course in physical statistics, but should be liberally supplemented with readings from the references which the author supplies at frequent intervals.

S. Prager (Brussels)

2887:

Johnson, D. E.; and Ikenberry, E. Developments toward a series solution of the Maxwell-Boltzmann equation. Arch. Rational Mech. Anal. **2** (1958), 41–65.

Linearize the collision integral in the Maxwell-Boltzmann integro-differential equation. The resulting linearized collision operator defines an eigenvalue problem. If the particles described interact with a repulsive inverse fifth power force (Maxwellian molecules), the eigenfunctions can be found [C. S. Wang Chang, G. E. Uhlenbeck: On the propagation of sound in monatonic gases. Univ. of Michigan Engineering Res. Inst. Rep., 1952].

In this paper the following objectives are achieved: (a) the standard form of the linearized collision operator is obtained by a straight-forward method; (b) the complete set of eigenfunctions of the operator for Maxwellian molecules is rederived; (c) expanding the distribution function in terms of the eigenfunctions, an infinite set of partial differential equations is obtained for the expansion coefficients which are formally equivalent with the original Boltzmann equation for Maxwellian molecules; this is given without external forces acting or with an external Lorentz force being present; (d) some of the expansion coefficients are expressed in terms of the pressure tensor, energy flux, density and velocity.

N. L. Balazs (Chicago, Ill.)

2888:

Bazarov, I. P. Equations with variational derivatives in statistical equilibrium theory. Soviet Physics. JETP **5** (1957), 872–882.

The method of functional derivatives, as introduced in statistical mechanics by N. N. Bogoliubov [Problems of dynamical theory in statistical physics, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1946 and Vestnik Moscow State University **10** no. 4–5, (1955), 115–124; MR **13**, 196; **17**, 222], is applied to the calculation of distribution functions in classical gases for Coulomb interaction, short range interaction and a superposition of these types of forces. The approximations used for Coulomb and short range forces are the usual low density ones. The case of superposition of Coulomb and short range forces is treated very briefly, the principle being an expansion in powers of $\exp(-V/kT)-1$, where V is the short range part of the potential. *L. Van Hove* (Utrecht)

2889:

Pütter, Paul Stefan. Zur Statistik der Systeme gleicher Teilchen: Die exakten dynamischen Grundgleichungen der klassischen Statistik. Ann. Physik (6) **19** (1956), 145–165.

2890:

Pütter, Paul Stefan. Anwendung der dynamischen Grundgleichungen der klassischen Statistik. Ann. Physik (6) **19** (1956), 247–256.

2891:

Pütter, Paul Stefan; und Sauter, Fritz. Zur Statistik des Plasmas. Die dynamischen Grundgleichungen einer klassischen Statistik des Plasmas. Ann. Physik (7) **1** (1958), 4–15.

Das von Born und Green zur Behandlung der Frage des flüssigen Zustandes entwickelte statistische Verfahren [Vgl. z.B. M. Born und H. S. Green, A general kinetic theory of liquids, Cambridge Univ. Press, 1949; MR **12**, 230] wird hier auf das Problem eines Plasmas angewendet. Schon in zwei vorangehenden Arbeiten hat einer der Verfasser diese Methode weiter entwickelt. [Vgl. P. S. Pütter, #2889 und #2890, oben stehend.]

Es wird ein nur teilweise ionisiertes Plasma betrachtet, das also aus Elektronen, einfach geladenen positiven Ionen und neutralen Teilchen, die kurz als Atome bezeichnet werden, besteht. Alle Teilchen werden als kugelsymmetrisch angenommen und es wird nur die klassische Theorie benutzt; Ionisations- und Rekombinationseffekte bleiben also unberücksichtigt. Damit die Coulombanziehung zwischen Elektronen und Ionen demzufolge in der Statistik keine Schwierigkeiten verursacht, wird noch eine fiktive Abstossungskraft zwischen diesen Teilchen eingeführt. Die von den geladenen Teilchen mitgeführten Felder werden nicht komplett mit Hilfe von Wiechert-Potentialen beschrieben, sondern es werden bloss Glieder bis zur ersten Potenz von $1/c$ berücksichtigt. Zuerst werden die äusseren Kräfte, also elektromagnetisches Vakuumfeld und sonstige Kräfte (Gravitation, Wandeinflüsse usw.) besprochen und dann die zwischen den einzelnen Teilchen wirkenden Kräfte. Danach wird die Gesamtkraft, die auf ein Teilchen (Elektron, Ion, Atom) einwirkt, angegeben. Diese rührt von den äusseren Kräften, von den anderen Teilchen — deren Orte bekannt sind — verursachten Kräften und von den bloss statistisch berücksichtigten übrigen Teilchen her. Mit Hilfe der für die Verteilungsfunktion schon hergeleiteten Rekursionsformeln werden dann solche Formeln für diese Kräfte hergeleitet und endlich werden die Differentialgleichungen für die Verteilungsfunktionen angegeben und deren Gültigkeit wird bewiesen. Diese Systeme von Differentialgleichungen können nur schrittweise und annähernd gelöst werden. Die Kontinuitätsgleichung und die Impulsgleichung werden aus den erhaltenen Formeln hergeleitet, ausserdem auch die erste Maxwellsche Gleichung, die zweite ($\partial\mathbf{H}/\partial t=-c \operatorname{rot} \mathbf{E}$) gilt dagegen bei den hier gemachten Annahmen nur annähernd. *T. Neugebauer* (Budapest)

2892:
Kohn, W.; and Luttinger, J. M. Quantum theory of electrical transport phenomena. Phys. Rev. (2) **108** (1957), 590–611.

The authors treat the theory of electrical transport quantum mechanically from first principles: they obtain the equations of motion for the density matrix for a system of non-interacting electrons in an external electric field. The collision mechanism is provided by a set of fixed, but randomly spaced, impurities. The density matrix for the steady state solution is obtained as a power series expansion in the scattering potential strength. The usual Boltzmann transport equation, in which the change in time of the distribution function for the particle to be in a state is due to a sum of terms representing the effect of external fields and the gains and losses due to collision, is shown to result only as a first approximation for very weak or dilute scatterers. Higher order corrections arising from the non-vanishing of off-diagonal elements in the density matrix at higher densities are also obtained. In second order approximation, these corrections represent the effect on the transition probability of interference between the scattering from two impurities averaged over all possible configurations of the pair.

D. Falkoff (Waltham, Mass.)

2893:
Luttinger, J. M.; and Kohn, W. Quantum theory of electrical transport phenomena. II. Phys. Rev. (2) **109** (1958), 1892–1909.

In the preceding paper the treatment was based on an expansion in powers of the strength of the scattering potential. In this paper the transport equation is obtained for the same model in powers of the density of scatterers without the restriction of weak scattering potentials. The expansion involves scattering operators for single centers, pairs of centers, etc., in a manner in many ways analogous to the virial expansion of equilibrium properties. The lowest order terms yield the usual Boltzmann equation. The first correction, in density, to this equation is explicitly given. For the case of spherically symmetric scatterers the solutions of these equations are also obtained. (Author's abstract) *D. Falkoff* (Waltham, Mass.)

2894:
Greenwood, D. A. The Boltzmann equation in the theory of electrical conduction in metals. Proc. Phys. Soc. **71** (1958), 585–596.

Because electrical resistance in a crystal with impurities is an irreversible phenomenon, it can only be derived from the Schrödinger equation of the electrons if suitable randomness assumptions are made. This is usually done in the form of random phases of the wave function. It is shown here that it is also sufficient to assume random phases of the matrix elements of the perturbing potential, which amounts to assuming that the impurities are located at random positions. It is then possible to express the off-diagonal elements of the density matrix in terms of the diagonal elements, so that an equation for the latter is obtained. This equation can be solved for an electron gas that is initially in equilibrium and subsequently subjected to an external electric field, which is slowly applied. The solution is valid to second order in the scattering potential and to first order in the external field. {The same approach was used by W. Kohn and J. M. Luttinger in the papers reviewed above.}

N. G. van Kampen (Utrecht)

2895:
Cowan, Robert D.; and Kirkwood, John G. Quantum statistical theory of electron correlation. Phys. Rev. (2) **111** (1958), 1460–1466.

In this paper the authors calculate the thermodynamical properties of a system of point electrons embedded in a positively charged background. The method is as follows: First one calculates from Poisson's equation the average electrostatic potential at a distance r from any specific electron, assuming that each electron carries a fraction of its actual charge and the density of the electron at any point in the presence of this potential is that obtained in the Thomas-Fermi approximation. Finally, we compute the work done in charging each electron from zero to its final charge in the presence of this potential. This gives the Helmholtz free energy from which the thermodynamical properties can be computed. The numerical calculations were done on an IBM computer and show that at low temperatures the total energy, pressure and specific heat are in excellent agreement with the values obtained by quantum mechanical calculations.

N. L. Balazs (Chicago, Ill.)

2896:
Gombás, P. Zur statistischen Theorie komprimierter Atome. Acta Phys. Acad. Sci. Hungar. **8** (1958), 321–358. (Russian summary)

Es wird gezeigt, dass die statistische Theorie komprimierter Atome durch die Erweiterung mit der Weizsäckerschen Korrektion im Verhältnis zu den Thomas-Fermischen und Thomas-Fermi-Diracschen Theorien eine wesentliche Änderung aufweist. Bei zunehmender Kompression steigt nämlich die Energie des Atoms nicht monoton an wie in den Thomas-Fermischen und Thomas-Fermi-Diracschen Theorien, sondern sinkt zunächst ab, durchläuft ein Minimum und steigt von dort an sehr steil an. Auf Grund dieser Resultate sind die aus den Thomas-Fermischen und Thomas-Fermi-Diracschen Modellen hergeleiteten Druck—Dichte Beziehungen einer Revision zu unterziehen. *Zusammenfassung des Autors*

2897:
Gombás, P. Störungsrechnung für das erweiterte statistische Atommodell. Acta Phys. Acad. Sci. Hungar. **8** (1958), 305–314. (Russian summary)

Es wird für das mit der ursprünglichen Weizsäckerschen Korrektion erweiterte statistische Atommodell, sowie für das mit der vom Verfasser hergeleiteten kinetischen Energiekorrektion modifizierte Modell eine Störungsrechnung entwickelt. Für die Störungsenergie erster und zweiter Ordnung, sowie für die gestörte Elektronendichte bis zu Glieder erster Ordnung werden die entsprechenden Ausdrücke explicite angegeben.

Zusammenfassung des Autors

2898:
Jaynes, E. T. Information theory and statistical mechanics. II. Phys. Rev. (2) **108** (1957), 171–190.

In an earlier paper [Phys. Rev. (2) **106** (1957), 620–630; MR **19**, 335] the author gave a presentation of equilibrium statistical mechanics (ensemble theory) based on Shannon's theorem in information theory concerning the entropy as information measure. Using the density matrix this approach is now extended to the case of non-equilibrium processes. The relation between irreversibility and loss of information is discussed in a semiclassical approximation. *L. Van Hove* (Utrecht)

2899:

Lebowitz, Joel L.; and Bergmann, Peter G. Irreversible Gibbsian ensembles. Ann. Physics 1 (1957), 1–23.

This continues an earlier investigation of the authors [Phys. Rev. (2) 99 (1955), 578–587; MR 17, 567] in which irreversible processes are described in terms of the interaction of the system studied with "driving reservoirs", each of which is at a definite temperature. The state distribution of the system is shown to approach a unique stationary state, independent of the initial conditions, and different from equilibrium whenever the reservoirs are not all at the same temperature. Reciprocity relations of the Onsager type are derived. *L. Van Hove* (Utrecht)

2900:

Davies, R. O. A note on the systematic integration of Kramer's equation for Brownian motion in a field of force. Physica 23 (1957), 1067–1068.

Kramer's equation for Brownian motion in a field of force [H. A. Kramers, Physica 7 (1940), 284–304; MR 2, 140] is solved by a series expansion method using a Laplace transform in time and the moments of the velocity distribution. *L. Van Hove* (Utrecht)

2901:

Szaniawski, Andrzej. Thermodynamics of irreversible phase change processes. Arch. Mech. Stos. 10 (1958), 399–416. (Polish and Russian summaries)

The author applies a power series development around the equilibrium values to get relations between thermodynamic variables for small irreversible changes for a system of homogeneous liquid and vapor phases in contact. *D. Falkoff* (Waltham, Mass.)

2902:

Trlifaj, Ladislav. Some aspects of the spherical harmonics method for neutron-transport problems in cylindrical geometry. Czechoslovak J. Phys. 8 (1958), 390–395. (Russian summary)

A solution is given of the Boltzmann differential equation for the transport of monoenergetic neutrons in a complete cylindrically symmetrical medium, which scatters neutrons isotropically, according to the method of spherical harmonics. The constants of integration are then expressed by means of the terms of the spherical harmonics moments. *From the author's summary*

ELASTICITY, PLASTICITY

2903:

✱Reissner, Eric. On variational principles in elasticity. Calculus of variations and its applications. Proceedings of Symposia in Applied Mathematics, Vol. VIII, pp. 1–6. McGraw-Hill Book Co., Inc., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1958. 153 pp. $7.50.

It is shown that the general linear boundary value problem of classical elasticity may be written as the Euler equation of a variational problem $\delta I=0$, where I is the sum of a volume and a surface integral whose integrands are quadratic polynomials in the displacements, displacement gradients, and stresses. The displacements and stresses are considered to be independent and unconstrained.

Under suitable constraints on the displacements, this variational principle reduces to Green's minimum energy principle. Under other constraints on the stresses, it reduces to Castigliano's maximum principle. In general, I is neither maximized nor minimized by the solution. *H. F. Weinberger* (College Park, Md.)

2904:

✱Reiner, M. Rheology. Handbuch der Physik, herausgegeben von S. Flügge. Bd. 6. Elastizität und Plastizität, pp. 434–550. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. DM 145.00.

This is an extremely valuable authoritative account of all aspects of rheology, and a masterly systematization of the most diverse phenomena. Mathematics is kept at an elementary level and there is a nice balance with experimental data. Different viewpoints and the current position are fairly presented. The main sections are (i) classical bodies, viz. the perfect elastic solid, Newtonian liquid, and Prandtl-Reuss plastic solid; (ii) macrorheology, which is essentially a classification in terms of combinations of the above basic elements; (iii) microrheology, which has to do with dispersed systems of two or more phases.

Non-linear effects are also thoroughly discussed, for instance rubberlike elasticity, structural viscosity, and cross-stress effects (Poynting and Weissenberg). The article closes with an account of rheometry. *R. Hill* (Nottingham)

2905:

Anglès d'Auriac, Paul. Formules générales de l'élasticité finie. C. R. Acad. Sci. Paris 246 (1958), 1962–1964.

The author writes equations of nonlinear elasticity theory using novel notation. *J. L. Ericksen* (Baltimore, Md.)

2906:

Anglès d'Auriac, Paul. Equations de l'élasticité finie du premier ordre. C. R. Acad. Sci. Paris 246 (1958), 2101–2103.

The author writes equations of linear and slightly nonlinear elasticity using novel notation. *J. L. Ericksen* (Baltimore, Md.)

2907:

Anglès d'Auriac, Paul. Cas particuliers de l'élasticité finie. C. R. Acad. Sci. Paris 246 (1958), 2217–2218.

The author discusses nonlinear elastic materials for which the strain energy is a sum of a part depending only on volume change and a part depending only on change of shape. Stronger results have been obtained by Richter [Z. Angew. Math. Mech. 28 (1948), 205–209; MR 10, 167]. *J. L. Ericksen* (Baltimore, Md.)

2908:

Duffy, J.; and Mindlin, R. D. Stress-strain relations and vibrations of a granular medium. J. Appl. Mech. 24 (1957), 585–593.

A differential stress-strain relation is derived for a medium composed of a face-centered cubic array of elastic spheres in contact. The stress-strain relation is based on the theory of elastic bodies in contact, and includes the effects of both normal and tangential components of contact forces. A description is given of an experiment performed as a test of the contact theories and the differential stress-strain relation derived from them. The experiment consists of a determination of wave velocities and the accompanying rates of energy dissipation in granular bars composed of face-centered cubic arrays of spheres. Experimental results indicate a close agreement between the theoretical and experimental

values of wave velocity. However, as in previous experiments with single contacts, the rate of energy dissipation is found to be proportional to the square of the maximum tangential contact force rather than to the cube, as predicted by the theory for small amplitudes. (Authors' summary.) *R. Hill* (Nottingham)

2909:

✱Deresiewicz, H. Mechanics of granular matter. Advances in applied mechanics, Vol. V, pp. 233–306. Academic Press Inc., New York, N. Y., 1958. x+459 pp. $12.00.

This is an admirable and well-documented review of theoretical and experimental work in a comparatively new branch of the mechanics of solids. The first part deals in great detail with the packing of equal and unequal spheres and also of non-spherical bodies. The next part is largely concerned with the Cattaneo-Mindlin theory of elastic contact of spheres under oblique forces, and with elaborations of this in, for instance, calculating hysteresis loops for oscillating tangential force when local slip occurs under the Coulomb friction law. In the final part there are brief accounts of wave propagation through regular arrays of equal spheres assuming only normal reactions [T. Takahashi and Y. Sato, Bull. Earthquake Res. Inst. Tokyo **27** (1949), 11–16; MR **13**, 185; F. Gassman, Vierteljschr. Naturf. Ges. Zürich **96** (1951), 1–23; MR **13**, 303], and through an irregular assembly of unequal spheres [H. Brandt, J. Appl. Mech. **22** (1955), 479–486]. The quantitative correlation of these theories with experiment is not good, but can be improved by taking account of tangential forces and twisting moments [Duffy and Mindlin, reviewed above; Thurston and Deresiewicz, in preparation]. The article closes with specific suggestions for further research. *R. Hill* (Nottingham)

2910:

Tolokonnikov, L. A. Equations of the non-linear theory of elasticity in terms of displacements. Prikl. Mat. Meh. 21 (1957), 815–822. (Russian)

In the general non-linear theory of homogeneous isotropic elastic bodies, the stress is an isotropic function of a tensor measure of finite deformation. One such measure is the tensor e_{ik} given in terms of the displacement gradients by

$$e_{ik}=\tfrac{1}{2}\left(\frac{\partial u_k}{\partial X_i}+\frac{\partial u_i}{\partial X_k}+\frac{\partial u_s}{\partial X_i}\frac{\partial u_s}{\partial X_k}\right),$$

where the X_i are initial rectangular Cartesian (Lagrange) coordinates of a material point. As a result of the assumption of homogeneity and isotropy, the scalar invariants of the stress are functions of three functionally independent scalar invariants of e_{ik}. The choice of a functionally independent set is arbitrary, and most of the complicated formulae of this paper are simply systems of relations between various independent sets of stress invariants and deformation invariants. The functional relations between stress invariants and deformation invariants are restricted by the existence of an elastic energy function from which the stresses are derived by differentiation [See Truesdell, J. Rational Mech. Anal. **1** (1952), 125–300; MR **13**, 794; § 41]. An open problem in finite elasticity theory is the formulation of additional restrictions on these functional relations between stress and deformation invariants which will rule out insensible behavior. The main result of this paper is the formulation of some specific restrictions of this type which the author supports by limited experimental data on a specific material. *R. A. Toupin* (Washington, D.C.)

2911:

Szelagowski, Franciszek. A rotating disc with a rigid circular inclusion at the centre. Arch. Mech. Stos. **10** (1958), 155–161. (Polish and Russian summaries)

This paper deals, by means of a method previously developed by the author [Bull. Acad. Polon. Sci. Cl. IV **4** (1956), 105–110], with the stresses in a rotating disc with a rigid circular inclusion. Formulas for the stresses are obtained by superposing solutions of simpler problems. For small inclusions in a steel disc, the author finds that the radial stress at the edge of the inclusion is about 50% higher than in a homogeneous disc. *W. E. Boyce* (Troy, N.Y.)

2912:

Chakravorty, J. G. Torsion of a conical bar of transversely isotropic material. Bull. Calcutta Math. Soc. **49** (1957), 29–32.

2913:

✱Goulard, Madeline; Lo, Hsu; and Bollard, R. J. H. Torsion with warping restraint of tapered beams. Proceedings of the Third Midwestern Conference on Solid Mechanics, 1957, pp. 100–112. University of Michigan Press, Ann Arbor, Mich., 1957. vi+250 pp. $5.50.

The case of a rectangular cross section tube in the shape of a truncated cone, with spanwise wall thickness variation a power of the spanwise coordinate, is solved through the use of differential equations ascribed to Argyris and Dunn [J. Roy. Aero. Soc. **51** (1947), 199–269] and also through a minimum energy approximation in Ritz fashion to the solution of the boundary value problem.

E. Reissner (Cambridge, Mass.)

2914:

Mechovrišvili, Š. S. Problems of the momentless strained state of a toroidal shell. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **24** (1957), 179–193. (Russian)

Stress systems are found for a toroidal shell of circular cross-section under certain given axisymmetric distributions of external forces. Some discussion of general solutions is given, the shells being of either circular or elliptic cross-section. *R. C. T. Smith* (Armidale)

2915:

Nash, William A.; and Sheng, P. L. An interation method for solving linear problems in the theory of shallow shells. J. Aero. Sci. **25** (1958), 267.

It is shown that the two differential equations of the linear theory of shallow shells for deflection w and stress function F, for a shell with middle surface equation $z/b=\lambda f(x,y)$, have solutions of the form $w=w_1+\lambda w_2+\cdots$, $F=F_1+\lambda F_2+\cdots$, where the successive differential equations for w_n and F_n are uncoupled equations of the flat-plate type, with right hand sides depending on F_{n-1} and w_{n-1}, respectively. {It would seem that the usefulness of this procedure requires that only such problems be treated by it for which the qualitative behavior of the shell is that of a flat plate.}

E. Reissner (Cambridge, Mass.)

2916:

Morozova, E. A. Mathematical foundation of the impossibility of computation for a toroidal shell according to the momentless theory. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 5, 7–12. (Russian)

2917:

Bassali, W. A. Transverse bending of infinite and semi-infinite thin elastic plates. II. Bull. Calcutta Math. Soc. **49** (1957), 119–127.

2918:

Özden, Kemal. Ein Beitrag zur Schalentheorie. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **21** (1956), 201–238 (1957). (Turkish summary)

An attempt is made to establish a system of shell equations by expanding the solutions of the field equation $\Delta\mathbf{u}+k$ grad div $\mathbf{u}=0$ in the form $\mathbf{u}=\sum \varepsilon^n\mathbf{u}_n$, where $\varepsilon=h/L$, h is of the order of the shell thickness and L is a representative linear dimension of the middle surface of the shell. The procedure, which involves copious use of differential operator matrices, is not pursued to the point at which a system of shell differential equations would appear which might be compared with known versions of such a system. However, explicit results are represented for a rotationally symmetric circular plate problem and for a symmetrical conical shell problem. In the latter problem one misses a formulation of the boundary conditions along the edges of the shell and also an appearance of boundary layer type contributions to the solution.

E. Reissner (Cambridge, Mass.)

2919:

Truesdell, C. General solution for the stresses in a curved membrane. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 1070–1072.

It is indicated in which way the principle of virtual work, into which compatibility equations for displacements (and their derivatives) are introduced by means of the Lagrange multiplier method, leads to stress function representations for stresses, the multiplier(s) assuming the role of stress function(s).

E. Reissner (Cambridge, Mass.)

2920:

De Silva, C. Nevin. Deformation of elastic paraboloidal shells of revolution. J. Appl. Mech. **24** (1957), 397–404.

The equations developed by Naghdi for the elastic deformation of shells of revolution [Quart. Appl. Math. **15** (1957), 41–52; under review for MR] are solved for the case of an axisymmetrically loaded paraboloidal shell of constant thickness. The equations consider the effect of transverse shear deformation. A solution that is valid at the singularity at the shell apex is obtained using the method of asymptotic integration due to Langer [Trans. Amer. Math. Soc. **33** (1931), 23–64; **37** (1935), 397–416]. The results of a numerical example indicate that the maximum bending stress at the apex is highly dependent upon shear deformation effects if the loading is limited to a narrow region about the apex.

S. R. Bodner (Providence, R.I.)

2921:

Borş, C. I. L'étude des équations de la statique des corps anisotropes, à l'aide des transformations de Fourier. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **7** (1956), no. 2, 99–106. (Romanian. Russian and French summaries)

Lehnickiĭ [Theory of elasticity of an anisotropic body, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1950; MR **13**, 182], for anisotropic bodies in equilibrium with stresses independent of z, has introduced stress functions and given the equations they satisfy. The author applies Fourier transforms to these equations and, in particular, considers stress distributions in a half space.

R. C. T. Smith (Armidale)

2922a:

Borş, C. I. La torsion, l'allongement et la flexion des barres cylindriques formées de plusieurs matériaux anisotropes. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **8** (1957), no. 2, 163–190. (Romanian. Russian and French summaries)

2922b:

Borş, C. I. La torsion, l'allongement et la flexion des barres orthotropes, formées du plusieurs matériaux. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **7** (1956), no. 2, 33–73. (Romanian. Russian and French summaries)

Theory of the torsion and bending of composite bars due to Mushelišvili [Some basic problems of the mathematical theory of elasticity, Noordhoff, Groningen, 1953; MR **15**, 370] and others is extended to the case where the different materials have a plane of elastic symmetry perpendicular to the axis of the bar but are not isotropic. The earlier article considers orthotropic materials only.

R. C. T. Smith (Armidale)

2923:

Payne, L. E. Inequalitites for eigenvalues of supported and free plates. Quart. Appl. Math. **16** (1958), 111–120.

Let D be a two-dimensional domain bounded by a closed curve C. The eigenvalues of the following problems in D, all arranged in non-decreasing order, are considered: the vibrating fixed membrane λ_n; the vibrating free membrane μ_n; the buckling simply supported plate Λ_n; the vibrating simply supported plate Ω_n^2; the buckling free plate Γ_n; and the vibrating free plate γ_n^2.

The following inequalities are shown to hold between these quantities and Poisson's ratio σ:

$$\Lambda_n\geqq(1-\sigma)\mu_2+\sigma\lambda_n;\ \Lambda_n\geqq\tfrac{1}{2}(1+\sigma)\lambda_n;$$
$$\Lambda_n\geqq\Gamma_{n+3}\geqq(1-\sigma)\mu_{[(n+3)/2]};$$
$$\Omega_n^2\geqq\lambda_n\Lambda_1;\ \gamma_{n+2}^2\geqq\mu_n\Gamma_4\geqq(1-\sigma)\mu_n\mu_2.$$

Moreover, if D is convex the further inequalities

$$\Lambda_n\leqq\lambda_n;\ \Omega_n\leqq\lambda_n;\ \Gamma_n\leqq\mu_n;\ \gamma_n\leqq\mu_n,$$

are shown to hold.

The more general inequalities are also more important, since they provide lower bounds for the plate eigenvalues in terms of the more easily found eigenvalues of membranes. A numerical example for a vibrating square plate gives a lower bound within 15% of the correct value.

The inequalities are all proved by defining in an ingenious manner a comparison function of the minimum principle for one eigenvalue problem in terms of the eigenfunctions of a second problem, and extracting all possible advantage from Green's theorem.

H. F. Weinberger (College Park, Md.)

2924:

Szegö, G. Note to my paper "On membranes and plates". Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 314–316.

The author gave a proof [same Proc. **36** (1950), 210–216; MR **11**, 757] that, if the eigenfunction corresponding to the lowest frequency of a vibrating clamped plate does not change its sign, then this frequency is at least as great as that of the circular clamped plate of equal area.

In this note the author repairs a flaw discovered by P. R. Beesack in the earlier proof.

H. F. Weinberger (College Park, Md.)

2925:
✕Онианшвили, О. Д. Некоторые динамические задачи теории оболочек. [Oniašvili, O. D. Some dynamical problems in the theory of shells.] Izdat. Akad. Nauk. SSSR, Moscow, 1957. 195 pp. 9.35 rubles.

(1) Engineering "moment" theory of B. Z. Vlasov. (2) Vibrating thin shells. (3) Application of variational methods to the solution of boundary problems. (4) Dynamic stability of thin shells. (5) Non-linear problems. (6) Resistance of thin shells to horizontal loads. (7) Actual dynamic testing of curved thin shell structures.

R. C. T. Smith (Armidale)

2926:
Hu, Hai-Chang; and Shi, Po-Ming. On the equilibrium and stability of elastic thin-walled cylinders. Sci. Sinica **5** (1956), 185–204.

Differential equations are established for deformations of cylindrical shells under the restrictive assumption that each cross section of the shell translates and rotates as a rigid body, which is the customary assumption in certain structural engineering theories intending to account for such effects as shear lag and end restraint against warping in beam bending and torsion. An account is taken of the presence of an initial axial force and of initial bending couples, the equations of the paper are applicable to appropriate buckling problems. Sample solutions are presented for problems with support conditions which allow the use of simple trigonometric functions of the axial coordinate of the shell.

E. Reissner (Cambridge, Mass.)

2927:
Roseau, Maurice. Diffraction d'ondes élastiques planes dans un milieu homogène encastré suivant un demi-plan. C. R. Acad. Sci. Paris **245** (1957), 1888–1890.

This brief note formulates the problem of the title in the following way. As dependent quantities the author chooses the volumetric strain $\Theta(x, y)$ and the rotation $\Omega(x, y)$, each of which satisfies the reduced wave equation:

$$\Delta\Omega+\sigma^2\Omega=0,\ \Delta\Theta+\tau^2\Theta=0,$$

with σ and τ the propagation speeds of shear and compressional waves. An incoming plane wave is prescribed, and the diffracted components of Θ and Ω are subjected to radiation conditions of the Sommerfeld type with the appropriate velocities σ and τ. With the aid of appropriate Green's functions the author is able to obtain integral equations of the Wiener-Hopf type for Ω and Θ which are not coupled — and this fact is of capital importance. The solution of the integral equations is not discussed.

J. J. Stoker (White Plains, N.Y.)

2928:
Olszak, W.; and Murzewski, J. Elastic-plastic bending of non-homogeneous orthotropic circular plates. II. Arch. Mech. Stos. **9** (1957), 605–630. (Polish and Russian summaries)

The equations derived in Part I [same Arch. **9** (1957), 467–485; MR **19**, 791] are put in a form suitable for numerical analysis. Two examples are worked in detail: (i) a circular plate supported at its circumference and uniformly loaded over a concentric circular region, and (ii) an annular ring clamped at its outer edge and uniformly loaded by shearing forces around its inner edge.

R. Hill (Nottingham)

2929:
Grigolyuk, È. I. On the buckling of thin shells beyond the elastic limit. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 10, 3–11. (Russian)

The bulk of the paper is devoted to a survey, by no means exhaustive, of the literature of the plastic buckling of shells. The author also derives three sets of equilibrium equations for a shell of unspecified form. One set is based on the Hencky stress-strain relations and assumes complete incompressibility. The second is a generalization of the first, using the same type of stress-strain law, but requiring only plastic incompressibility. The third set is based on incremental theory and again requires only plastic incompressibility. No discussion is presented of the equations, nor is any attempt made to solve them.

P. Mann-Nachbar (Palo Alto, Calif.)

2930:
Grigolyuk, È. I. Loss of stability in ideally plastic thin shells. Prikl. Mat. Meh. **21** (1957), 846–849. (Russian)

This paper continues the work reviewed above on the stability theory of thin, ideally-plastic, shells under external loads. Herein considered are spherical shells under external pressure and cylindrical shells under axial compression, normal pressure and torsion. The analysis is based upon deformation plasticity theory and the shells are supposed ideal in shape. *H. G. Hopkins* (Sevenoaks)

2931:
Lepik, Yu. R. On stability of an elasto-plastic rectangular plate compressed in one direction. Prikl. Mat. Meh. **21** (1957), 722–724. (Russian)

An approximate solution by Galerkin's method; the boundary conditions are: loaded edges clamped, sides simply supported. *R. C. T. Smith* (Armidale)

STRUCTURE OF MATTER

See also 2862.

2932:
McLachlan, Dan, Jr. Crystal structure and information theory. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 948–956.

The author has introduced a novel and interesting way of looking at the theory of X-ray crystallography. Crystallographers have long assumed that, if the number of observed data exceeds the number of parameters to be determined by a factor of two or three, then the structure can be solved despite slight inaccuracies in the observed data. This proposition is proved by deriving expressions called the descriptive capacity of the crystal (W_c) and the descriptive capacity of the diffraction pattern (W_d). Then the condition for the structure to be soluble with the data available is given as $W_c \leq W_d$. In W_c and W_d are considered such things as the number of parameters involved, the number of data available, the estimated accuracy of the data, and the desired accuracy with which the atomic coordinates are to be determined.

By an optical analogy it is shown that the presence of a large number of identical unit cells in a given crystal greatly enhances the accuracy with which the structure can be determined. However, although this is true, the analogy cited is not exact.

{An unfortunate point is that the author has used $1/V$ (V=volume of unit cell) as the normalisation factor in a one dimensional Fourier series. This should be $1/a$

(a=unit cell edge).} *W. M. Macintyre* (Boulder, Colo.)

2933:

Harrison, Walter A. Cellular method for wave functions in imperfect metal lattices. Phys. Rev. (2) **110** (1958), 14–25.

The paper deals with a general method for constructing conduction band wave functions in nonperiodic monovalent metals and alloys. It is an improvement over previous schemes which were based on perturbations of periodic structures. The method consists of dividing the system into cells, determining a function for each cell and then matching the functions on the cellular boundaries. The individual functions are determined by the Wigner-Seitz cellular method [Wigner and Seitz, same Rev. **43** (1933), 804–810; **46** (1934), 509–524; J. Bardeen, J. Chem. Phys. **6** (1938), 367–378; Hunter and Nabarro, Proc. Roy. Soc. London Ser. A **220** (1953), 542–561]. Scattering by point defects is considered in detail, but dislocations, stacking faults, and lattice vibrations are also mentioned and various successes cited.

M. J. Moravcsik (Livermore, Calif.)

2934:

Miyakawa, Kozaburo. New derivation of elastic equations for trigonal holoaxial crystals. Phys. Rev. (2) **107** (1957), 677–682.

FLUID MECHANICS, ACOUSTICS

See also 2873, 2904, 2983, 2989, 3018, 3025, 3026, 3027.

2935:

Gheorghiev, Gh. Sur certains mouvements des fluides dont les lignes de courant sont isotachées. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **8** (1957), no. 2, 157–161. (Romanian. Russian and French summaries)

Certain geometrical results are derived from the continuity equation for stationary fluid motions for which the streamlines coincide with the curves along which the speed remains constant. Particular attention is paid to ideal barotropic fluids. Results of F. Sbrana [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (6) **7** (1928), 641–643], L. Castoldi [ibid. (8) **3** (1947), 333–337; MR **9**, 630] and S. S. Byušgens [Izv. Akad. Nauk SSSR. Ser. Mat. **12** (1948), 481–512; MR **10**, 633] are taken into consideration. *D. J. Struik* (Cambridge, Mass.)

2936:

Gillis, J.; and Goldstine, H. H. The Taylor problem for superposed fluids. Rend. Circ. Mat. Palermo (2) **6** (1957), 83–102.

The horizontal interface between two homogeneous, incompressible, non-viscous fluids at rest under gravity is subjected to a velocity perturbation of the form $\alpha \cos kx$. An analytic representation of the subsequent motion correct up to terms $O(\alpha^5)$ is obtained. Numerical results are given in the special case that the density of the lower fluid is negligibly small. *R. C. DiPrima* (Troy,N.Y.)

2937:

Leruste, Philippe. Representation Lagrangienne d'un fluide parfait. Acta Phys. Polon. **17** (1958), 3–12.

2938:

Stewartson, K. On the motion of a sphere along the axis of a rotating fluid. Quart. J. Mech. Appl. Math. **11** (1958), 39–51.

A sphere of radius a is assumed to move with uniform velocity W in perfect fluid which is rotating with constant angular velocity Ω; the centre of the sphere moves along the axis of rotation of the fluid, and the character of the motion is then determined by the Rossby number ka, where $\Omega=\frac{1}{2}kW$. The stream function is expanded as a series; it is found that when $ka=5.76$ the coefficients of the series become infinite, if it is assumed that the motion of the fluid far upstream is undisturbed. It is concluded that, for this value of ka, there must be a region near the axis in which there is a cylindrical motion, i.e., in which the fluid in front of the sphere is pushed along bodily by the sphere. Such a cylindrical motion was found experimentally by G. I. Taylor [Proc. Roy. Soc. London Ser. A **102** (1922), 180–189; **104** (1923), 213–218] provided that ka was in excess of a critical value of approximate magnitude 2π; in the experiments the fluid was contained in a circular cylinder. A further conclusion of this paper is that there is probably cylindrical motion if $ka\geqq 3$, and possibly for all $ka>0$, although in the latter case there would be a discrepancy with observation; it is not thought that such a discrepancy could arise from the containing cylinder used in the experiments. *W. R. Dean* (London)

2939:

Carpenter, Lloyd H. On the motion of two cylinders in an ideal fluid. J. Res. Nat. Bur. Standards **61** (1958), 83–87.

The complex potential of two cylinders moving in an infinite liquid is determined by the method of image doublets, and the solution is expressed as an infinite series in rectangular co-ordinates. Approximate solutions in finite form are given for various cases. A method for generalizing the solution for the case of more than two cylinders is indicated. Applications to the flow induced by a cylinder moving in the presence of plane boundaries are given and the stream lines are illustrated in certain cases. (Author's summary) *D. W. Dunn* (Ottawa, Ont.)

2940:

Filimon, Ioan. Sur l'équation intégro-différentielle de Prandtl. Acad. R. P. Romîne Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 381–385. (Romanian. Russian and French summaries)

L'équation intégro-différentielle de Prandtl, concernant la détermination de la circulation autour d'un profil d'envergure finie, est équivalente à la résolution d'un problème aux limites mixtes ainsi que l'a démontré E. Trefftz.

En partant de cette dernière forme du problème de Prandtl, l'auteur démontre que pour les profils dont les paramètres de forme ont pour expression:

$$p(\sigma)=\frac{m_0+\sum_{k=1}^{k=s} m_{2k}\cos(2k\sigma)}{n_0+\sum_{k=1}^{k=l} n_{2k}\cos(2k\sigma)}$$

la détermination de la circulation se réduit à des quadratures. *Résumé de l'auteur*

2941:

Evangelisti, Giuseppe. Sopra la potenza erogata in moto vario da correnti liquide entro tubi elastici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 287–290.

2942:
Martinek, J.; Yeh, G. C. K.; and Zorn, H. Potential and stream function of a vortex disk in the presence of a rigid sphere. Proc. Cambridge Philos. Soc. **53** (1957), 717–727.

2943:
Bhattacharyya, R. N. Wave resistance of a ship moving in a circular path. Proc. Nat. Inst. Sci. India. Part A. **24** (1958), 45–54.

A Mitchell-type ship moves so that its center plane remains tangent to a circle. Assuming the ship can be replaced by a source distribution on the center plane, the resulting wave motion is determined and wave resistance computed. *R. C. MacCamy* (Pittsburgh, Pa.)

2944:
Bartholomeusz, E. F. The reflexion of long waves at a step. Proc. Cambridge Philos. Soc. **54** (1958), 106–118.

The problem, which refers to gravity waves in a canal with a wave length large compared to depth, is treated rigorously by means of a formulation in terms of a Fredholm integral equation which can be solved by an iterative procedure when the wave length is large enough. The details of the analysis are interesting, but too intricate to be described in a brief review. The author finds, as a concrete result, that the reflection coefficient given by Lamb for a step in the bed of the canal, and obtained by him from the linearized version of the shallow water theory, is asymptotically correct. *J. J. Stoker* (White Plains, N.Y.)

2945:
Carrier, G. F.; and Greenspan, H. P. Water waves of finite amplitude on a sloping beach. J. Fluid Mech. **4** (1958), 97–109.

The problems are treated on the basis of the nonlinear shallow water theory. The differential equations, in dimensionless form, are

$$v_t+vv_x+\eta_x=0,$$
$$[v(\eta-x)]_x+\eta_t=0,$$

with $v(x, t)$ the horizontal velocity component and $\eta(x, t)$ the vertical displacement of the free surface; the slope of the beach is assumed constant. It is quite surprising (to this reviewer, at least) that simple, interesting, explicit solutions of these nonlinear partial differential equations have been found by the authors. One type of solution is periodic in time and represents a progressing wave reflected from the shore without breaking. Other solutions are for cases in which special initial conditions are prescribed at $t=0$, and again no breaking occurs. It had been generally believed that any wave for which an elevation of the free surface above the mean level occurs would always be found to break upon employing the theory under discussion here, since that is known always to occur in a canal of uniform depth when such a wave propagates into still water, and it is natural to expect that a decrease in depth to zero would serve to accentuate the effect. *J. J. Stoker* (White Plains, N.Y.)

2946:
Greenspan, H. P. On the breaking of water waves of finite amplitude on a sloping beach. J. Fluid Mech. **4** (1958), 330–334.

In the previous paper [#2945] examples of waves on a sloping beach are given which do not break as they approach the shore line. These waves were the result of special initial conditions. In the present paper the author shows that any wave of positive amplitude which progresses through still water toward shore will always break — as it would in water of constant depth — provided that the wave front has a nonzero slope. This is achieved by obtaining an explicit solution of the differential equations given for the preceding paper when the initial elevation of the free surface is arbitrarily prescribed. *J. J. Stoker* (White Plains, N.Y.)

2947:
Rott, Nicholas. On the viscous core of a line vortex. Z. Angew. Math. Phys. **9b** (1958), 543–553.

An exact solution of the equations of motion of an incompressible viscous liquid is shown to be

$$u=-ar,\ rv=b\{1-\exp(-ar^2/2\nu)\},\ w=2az,$$

where a, b are constants and u, v, w are the velocity components in cylindrical coordinates r, θ, z; there is accordingly no infinity at $r=0$. The result is applied to fluid motions corresponding to the "bathtub vortex" and to the tornado by restricting the range of the r and z coordinates. It is also shown that there are some exact solutions of the same type of the equations of unsteady motion. *W. R. Dean* (London)

2948:
Takaisi, Yorisaburo. The wall-effect upon the forces experienced by an elliptic cylinder in a viscous liquid. J. Phys. Soc. Japan **13** (1958), 496–506.

This is a calculation of the forces acting on an elliptic cylinder which is in steady two-dimensional motion in viscous liquid bounded by a single plane wall or by two parallel walls. The cylinder moves parallel to the wall; the angle between the axis of the cylinder and the wall is arbitrary, and a first approximation (appropriate to the case of slow motion) to the general solution of Oseen's equations is used. Computation of the lift and drag shows the effect on these components of the wall or walls in numerous cases, the effect on the lift being always the larger. *W. R. Dean* (London)

2949:
Kuwabara, Shinji. The forces experienced by two elliptic cylinders in a uniform flow at small Reynolds numbers. J. Phys. Soc. Japan **13** (1958), 506–519.

A first approximation (appropriate to the case of slow motion) to the general solution of Oseen's equations is used to determine the forces exerted on two elliptic cylinders in a steady two-dimensional stream of viscous liquid. Formulae are derived applicable in the general case of an arbitrary disposition of the axes of the cylinders and an arbitrary size and eccentricity of their cross-sections. In the cases in which the forces are computed, the plane of the two axes is either parallel to or perpendicular to the line of flows. *W. R. Dean* (London)

2950:
Spalding, D. B. Heat transfer from surfaces of non-uniform temperature. J. Fluid Mech. **4** (1958), 22–32.

This paper is concerned with heat transfer through a laminar boundary layer from the surface of a body immersed in a flow at low Mach number. The method of the reviewer [Proc. Roy. Soc. London Ser. A **202** (1950), 359–377; MR **12**, 218], which is asymptotically exact as the Prandtl number $\sigma\to\infty$, takes the velocity profile as linear within the thermal boundary layer. The present author writes the reviewer's result as

$$\frac{1}{\alpha}\left(\frac{\delta_4}{u_1}\right)^{\frac{1}{2}}\frac{d}{dx}\left[\Delta_4^3\left(\frac{u_1}{\delta_4}\right)^{\frac{3}{2}}\right]=6.41,$$

where α=thermal diffusivity, u_1=external flow velocity, δ_4="shear thickness"$=u_1/(\partial u/\partial y)_{y=0}$ and Δ_4="conduction thickness" $=-T_0/(\partial T/\partial y)_{y=0}$, where T_0 is the value at the wall of the temperature excess T over the stream temperature. He then adds a correction

$$F\left(\frac{\Delta_4\delta_4}{\nu}\frac{du_1}{dx}\right)$$

to the right hand side; the argument of this function is twice the ratio of the quadratic and linear terms in the expansion of u in powers of y at the point $y=\Delta_4$. The function F is taken as the smooth curve which best fits data accurately calculated by a large number of writers. The local heat transfer distribution is then easily deduced, probably to good accuracy, by numerically solving the differential equation.

For obtaining total heat transfer over a stretch of the surface, the author recommends a slightly different approach, in which the "enthalpy flux thickness" Δ_2 replaces Δ_4 and a different correction function is used.

M. J. Lighthill (Manchester)

2951:

Gibellato, Silvio. Strato limite termico attorno a una lastra piana investita da una corrente lievemente pulsante di fluido incompressibile. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **91** (1956–57), 152–170.

Following an earlier paper [same Atti, **89** (1955), 180–192; MR **17**, 205] on the boundary layer on a semi-infinite flat plate in the two-dimensional flow parallel to the plate of a uniform stream with velocity $U_\infty+u_\infty\cos\omega t$, with $u_\infty<<U_\infty$, the author determines the heat transfer from the plate, supposedly maintained at a constant excess of temperature over that of the stream, in the form of a series in ascending powers of the frequency ω. Terms in ω and ω^2 in the heat transfer and the temperature profile are determined for $\sigma=1$ and $\sigma=0.7$ where σ is the Prandtl number, and the convergence of the series is proved for $\sigma=1$. The coefficient of ω in the heat transfer is very small, as the reviewer obtained [Proc. Roy. Soc. London Ser. A **224** (1954), 1–23; MR **15**, 907] by an approximate method, but that of ω^2 is considerably larger.

M. J. Lighthill (Manchester)

2952:

Bourne, D. E.; and Davies, D. R. On the calculation of eddy viscosity and of heat transfer in a turbulent boundary layer on a flat surface. Quart. J. Mech. Appl. Math. **11** (1958), 223–234.

The distribution of eddy viscosity in the turbulent boundary layer on a flat plate can be calculated if the velocity profiles are known. This is not, as suggested, a new idea; for instance, equation (16), which gives the value in the turbulent core, was derived in Aero. Res. Council (G.B.) Rep. no. 14, 162 (1951). For the outer part of the layer the authors use similarity profiles for velocity based on $\xi=y/kx^q$ ($q\doteqdot 0.10$), which is justifiable for Reynolds numbers R_x of the order of 10^6, to obtain a distribution of eddy viscosity through the whole of the layer. Reynolds analogy in its strict form is then used to provide estimates of the heat transfer to a constant temperature wall. It would be interesting to see this analysis carried out for a turbulent Prandtl number different from unity, as has been done by Van Driest using experimental distributions of eddy viscosity; improved agreement with observed heat transfer coefficients might then have been found.

D. A. Spence (Farnborough)

2953:

Esch, Robin E. The instability of a shear layer between two parallel streams. J. Fluid Mech. 3 (1957), 289–303.

An unbounded parallel flow, consisting of a linear shear layer between uniform streams, is investigated for its stability characteristics. The most remarkable feature is that unstable disturbances are found at all values of the Reynolds number R. Indeed, for small values of R, the neutral curve takes on the form $\alpha=0.380(\alpha R)^{\frac{1}{3}}$. The complete neutral curve and some typical amplification rates are also determined by analytical and numerical methods. The extent to which the present results can be applied to the laminar shear between free streams is discussed.

C. C. Lin (Cambridge, Mass.)

2954:

Drazin, P. G. The stability of a shear layer in an unbounded heterogeneous inviscid fluid. J. Fluid Mech. **4** (1958), 214–224.

Consider a parallel, horizontal flow of an incompressible, inviscid fluid whose velocity is $U=V\tanh(y/d)$, where V and d are constants and y is the vertical coordinate. Assume that the density varies as $\exp(-\beta y)$. The problem studied is the stability of this flow with respect to small disturbances when the effect of gravity is taken into account. S. Goldstein [Proc. Roy. Soc. London Ser. A **132** (1931), 524–548] studied a similar problem for a certain particular flow with discontinuous gradients of velocity and density. For the flow discussed by the author the problem reduces, with the help of some simplifying physical assumptions, to the eigenvalue problem $d^2\phi/dy^2+(2\sec h^2y-a^2+J\cot h^2y)\phi=0$; $a\phi\to 0$, as $y\to\pm\infty$, where the parameters a and J are the wave number of a periodic disturbance and Richardson's number, respectively. By means of the theory of Fuchsian differential equations, the eigenvalue equation is found to be $J=a^2(1-a^2)$, and it follows that the flow is stable if $J>\frac{1}{4}$. Since this is the same result as Goldstein's, the author considers it a plausible inference that discontinuities in the gradient of velocity or density do not influence the stability of such flows decisively.

W. Wasow (Madison, Wis.)

2955:

Gribov, V. N.; and Gurevich, L. E. On the theory of the stability of a layer located at a superadiabatic temperature gradient in a gravitational field. Soviet Physics. JETP **4** (1957), 720–729.

Previously, Rayleigh [Philos. Mag. **32** (1916), 529–546], Pellew and Southwell [Proc. Roy. Soc. London. Ser. A **176** (1940), 312–343; MR **2**, 266] and Chandrasekhar [ibid. **217** (1953), 306–327; MR **15**, 174] investigated the stability of a horizontal layer of fluid, in a gravitational field, which is bounded on one or both sides by stable layers in which the temperature gradient is less than the adiabatic value. The present paper considers the propagation of the convective flow, which has been generated inside the unstable layer and has obtained momentum as it rises toward the upper boundary of the unstable layer, into the stable region. Two different cases are analyzed separately: (1) convection may propagate only upward; and (2) convection may propagate both upward and downward from the unstable layer. In each case the effective distance penetrated by the convective flow into

the stable region is estimated under more specific conditions. *T. Yao-tsu-Wu* (Pasadena, Calif.)

2956:

Broer, L. J. F. Characteristics of the equations of motion of a reacting gas. J. Fluid Mech. **4** (1958), 276–282.

The non-uniform behaviour of the characteristic speed of a chemically reacting gas without viscosity or heat conduction is discussed. The phase velocity of small disturbances is calculated for the case of a single reaction and it is pointed out that in general the phase velocity is complex. In the limit case of very high frequencies, the usual speed of sound for fixed chemical composition holds. It is shown that the characteristics of the partial differential equations correspond to propagation at infinitely high frequency. But it is also shown that the characteristic speed does not change continuously from the high frequency to the low frequency value as the reaction rate is made infinitely large in motion near the equilibrium state. Instead, the propagation speed of discontinuities always remains at the high frequency value until the limit case of infinite rate is reached. The author points out that the true characteristic equations of these flows involve only the high frequency propagation limit except for the special case of infinite reaction rate. At finite reaction rates characteristic-like equations which are useful for computation can be written, but these may lead to spurious theoretical conclusions.

Similar discussions have been given by the author [Appl. Sci. Res. A. **2** (1950), 447–468; MR **12**, 767], W. Lick [Rensselaer Polytech. Inst. TR AE 5810, AD no. 158335] and Wood and Kirkwood [J. Appl. Phys. **28** (1957), 395–398]. *Hirsh Cohen* (Delft)

2957:

Resler, E. L., Jr. Characteristics and sound speed in nonisentropic gas flows with nonequilibrium thermodynamic states. J. Aero. Sci. **24** (1957), 785–790.

It is the intention of this paper to develop methods of handling gas flows with chemical reactions. In order to do this the sound speed is defined and characteristic equations are written down. In the light of the discussion given by Broer [#2956 above] and Kirkwood and Wood [J. Appl. Phys. **28** (1957), 395–398], it is not clear that the non-uniform behaviour of the characteristic speed is taken into account nor that the equations are indeed characteristic equations. The question of the phase velocity being complex and of its changing behavior at the low and high frequency limit is not discussed. On the basis of this, the discussion given for various cases of nonisentropic flows in non-equilibrium states must be given closer attention. It is possible that the sound-speed definition used is an appropriate algorithm for the cases considered in this paper. These include, for unsteady one-dimensional flow, the cases of a perfect gas with lagging heat capacity, a diatomic molecule gas with lagging rotational heat capacity, a diatomic molecule gas with lagging vibrational heat capacity and a dissociating diatomic gas with lagging vibrational heat capacity. Also included is a similar discussion for two dimensional steady flow for the cases of nonisentropic flow starting from uniform conditions and a dissociating gas with lagging vibrational heat capacity.

Hirsh Cohen (Delft)

2958:

Moore, F. K. Propagation of weak waves in a dissociated gas. J. Aero. Sci. **25** (1958), 279–280.

In this note it is shown that definitions for an equilibrium sound speed (low frequency limit) and for a frozen sound speed (high frequency limit) may be obtained from the linear differential equation of the propagation of weak disturbances. The linear distortion due to the complex phase velocity is observed. Broer [#2956 above], however, has pointed out that the low frequency sound speed defined here is a true wave propagation rate only for the case of infinite reaction rate, and that there is a jump from the high frequency speed to this speed in the limit case. *Hirsh Cohen* (Delft)

2959:

Fraser, A. R. Radiation fronts. Proc. Roy. Soc. London. Ser. A. **245** (1958), 536–545.

The author defines a radiation front as a thin region, separating hot and cold fluid, whose dynamics is affected by radiation incident on it from the side of the hot fluid. His analysis of radiation fronts as discontinuities, and of the structure of radiation fronts, is almost exactly the same as the classical analysis of exothermic discontinuities, such as deflagration and detonations [see for example Emmons (ed.), Fundamentals of gas dynamics, Princeton Univ. Press, 1958; MR **20** #3690; Sections D and G], since he makes no explicit assumption about the functional dependence of the heat input term (here, the radiation flux) on other parameters. His arguments follow all the normal lines, except that, surprisingly, he makes no conclusion regarding the special importance of the Chapman-Jouguet condition.

{This is a paper from a government establishment, and one expects that the rather trite conclusions could have been made much more interesting had the author been free to set down all he knew, particularly on the observational side and in connection with the magnitude of the radiation flux. It is doubtful whether anything is gained if publication is permitted only in such an emasculated form.} *M. J. Lighthill* (Manchester)

2960:

✱Bers, Lipman. Mathematical aspects of subsonic and transonic gas dynamics. Surveys in Applied Mathematics, Vol. 3. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1958. xv+164 pp. $7.75.

The physical problems of fluid dynamics have provided the stimulus for the initial development of important branches of pure mathematics, and it appears as if they will continue to do so. During the last two decades, an increasing number of mathematicians have been drawn to the study of questions suggested by problems of potential flow of compressible inviscid fluids. Professor Bers' book is a most welcome survey of the results accumulated to date by the rigorous applied mathematicians.

The book falls into two parts, because of the strong distinction between the nature of the mathematical aspects of subsonic and transonic problems. It is often thought that there is a radical difference between linear and non-linear partial differential equations, but the quasilinear equations of inviscid gas dynamics turn out to be topologically equivalent to the classical potential equation of Laplace, when only solutions in the elliptic domain are considered. Accordingly, the subsonic "non-

linear" problems have been found susceptible to treatment by elegant generalisations of classical concepts of analysis, such as quasi-conformal mapping and pseudo-analytic functions, and altogether a very quasi-beautiful branch of mathematics has been mapped out.

The same quasilinear equations are topologically different from the classical wave equation in the hyperbolic domain, but that is only one of several factors contributing to the serious and fundamental difficulties encountered in the study of their mixed elliptic-hyperbolic solutions. In fact, the corresponding linear equations pose similar problems, and most of the effort has rightly been poured into their study. Important progress has been made toward reducing the wide gaps in knowledge concerning existence and uniqueness, but meanwhile, the fact has emerged (and this is admirably brought out in this survey) that even existence is not a concept touching the root of the difficulty! The transonic part of the book thus presents a striking account of what may be conjectured to be birth pangs, still continuing, of a new branch of mathematics with potential influence on most other branches.

The survey is a thorough one, as witnessed by 400 references, brought nearly up to the date of publication. It should be said, even though this is taken for granted by everybody in the case of Prof. Bers, that the survey is masterly in its elegance and clarity. To achieve an enjoyable account covering so vast a body of complicated investigations required, beyond the author's own process of illumination and haute couture, a certain amount of choice favouring elegant papers and theories. It might be fair to warn the reader that the author abstains from offering very explicit help in distinguishing what is elegant and what has substance as well. In any case, the book will be nearly indispensable to all those connected with modern analysis and compressible flow theory.

R. E. Meyer (Providence, R.I.)

2961:

Morawetz, Cathleen S. On the non-existence of continuous transonic flows past profiles. III. Comm. Pure Appl. Math. **11** (1958), 129–144.

Il s'agit d'une nouvelle contribution de l'auteur à l'examen de l'existence d'écoulements transsoniques continus stationnaires autour de profils [Comm. Pure Appl. Math. **9** (1956), 45–68; **10** (1957), 107–113; MR **17**, 1149; **19**, 490].

L'objet du travail est le suivant: partant d'un écoulement transsonique continu, envisager si une perturbation de certaines conditions imposées à cet écoulement peut donner lieu à un nouvel écoulement; la méthode consiste toujours à linéariser le problème relativement à la solution donnée et à envisager la perturbation dans le plan de l'hodographe. On est ramené à l'examen d'un problème d'unicité. Dans la partie III dont il est question ici, est envisagée une perturbation de la vitesse à l'infini. Le résultat atteint est le suivant: considérons une famille d'équations d'état, définie par la relation $c(q)$ entre la célérité du son et la vitesse, tous les $c(q)$ étant identiques pour $q \leq \bar{q}$ ($\bar{q}$ supersonique), mais distincts, quoiqu'arbitrairement voisins pour $q \geq \bar{q}$, alors pour tous les écoulements continus correspondant à cette famille ayant même frontière dans le plan de l'hodographe pour $q \leq \bar{q}$, il en existe un au plus conduisant à une perturbation continue lorsqu'on modifie la vitesse à l'infini. Le théorème est démontré en mettant en évidence une contradiction, au cas où il y aurait deux tels écoulements, relative au comportement des lois d'état au voisinage de $q=\bar{q}$.

P. Germain (Paris)

2962:

Münch, Johann. Beiträge zum Quellsenkenverfahren für die Berechnung von Überschallströmungen. Z. Angew. Math. Mech. **37** (1957), 51–63. (English, French and Russian summaries)

2963:

Carafoli, Elie; and Horovitz, Béatrice. L'écoulement supersonique homogène, d'ordre supérieur, autour d'une aile angulaire à plaque normale. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **8** (1957), 959–974. (Romanian. Russian and French summaries)

Consider a body composed approximately of two sectors of the x_1x_2 and x_1x_3 planes, with common vertices at the origin. To find the linearized supersonic flow about it parallel to the x_1-axis with velocity distributions on wing and plate of a type described below, the author seeks a velocity potential $\Phi(x_1, x_2, x_3)$ that is homogeneous of order n in x_1, x_2, x_3. By Euler's formula the velocity components $u=\Phi_{100}$, $v=\Phi_{010}$, $w=\Phi_{001}$ are expressible as linear combination of $\Phi_{p,q,r}=\partial^n\Phi/\partial x_1{}^p\partial x_2{}^q\partial x_3{}^r$, $p+q+r=n$, with coefficients that are known functions of x_1, $y=x_2/x_1$, and $z=x_3/x_1$. On the wing, approximately in the x_1x_3 plane, for example, $(n-1)!w/x_1{}^{n-1}=\sum_0^{n-1} C_{n-1}{}^q z^q \Phi_{n-q-1,q,1}$ where $C_{n-1}{}^q$ is a binomial coefficient. To obtain normal velocity distributions of the type $w=\sum_0^{n-1} w_{n-q-1,q} x_1{}^{n-q} x_2{}^q$ on the wing, impose the boundary conditions $C_{n-1}{}^q\Phi_{n-q-1,q,1}=(n-1)!w_{n-q-1,q}$. Also, impose a homogeneous polynomial distribution of normal velocity on the plate in the x_1x_2-plane. As in conical flow, $\Phi_{p,q,r}$ are harmonic functions of certain distorted coordinates η, ζ in the yz-plane. Let $\Phi_{p,q,r}'$ be the conjugate harmonic functions, and let $F_{p,q,r}(\xi)=\Phi_{p,q,r}+\Phi_{p,q,r}'$, where $\xi=\eta+i\zeta$. By virtue of certain compatibility relations among $F_{p,q,r}$, it suffices to find only $F_{n,0,0}$, for which $\Phi_{n,0,0}=0$ on $|\xi|=1$ and $\Phi_{n,0,0}'=0$ on the segments of the real and imaginary axes that correspond to wing and plate. By considering the nature of the singularities expected at the leading edges of wing and plate, the author is able to determine $F_{n,0,0}$ explicitly.

J. H. Giese (Aberdeen, Md.)

2964:

Gazaryan, Yu. L. Sound field generated by a point source in a layer lying on a halfspace. Akust. Ž. **4** (1958), 233–238. (Russian)

The distant field of an acoustic point source in a homogeneous liquid layer with free upper boundary and lying on a homogeneous liquid half-space can be represented as the sum of normal modes and lateral waves [L. M. Brehovskih, Izv. Akad. Nauk SSSR Ser. Fiz. **13** (1949), 505–545; MR **11**, 563, 564]. Properties of the normal modes have been studied by C. L. Pekeris [Geol. Soc. of America Memoir **27** (1948)]. The present author studies the lateral waves, which arise from a branch-line integral in the contour integration for the velocity potential, when there is complete reflection in the layer. He obtains two asymptotic expansions for the integral in question, valid for frequencies which are, respectively, remote from and near to a critical frequency. Using the latter expansion, he discusses the behavior of the field at a fixed (large) distances as the frequency increases so as to pass through the critical frequency of one of the normal modes.

R. N. Goss (San Diego, Calif.)

2965:

Colombo, Serge. La théorie hydromagnétique. Cahiers de Phys. 92 (1958), 129–153.

L'auteur expose les bases de la magnéto-hydrodynamique et discute le domaine de validité des équations. La notion de nombre de Reynolds hydromagnétique permet de prévoir les conditions dans lesquelles les effets magnétiques sont prépondérants. Les ondes d'Alfven sont étudiées par la méthode de linéarisation.

H. Cabannes (Marseille)

2966:

Bleviss, Z. O. Magnetogasdynamics of hypersonic Couette flow. J. Aero. Sci. 25 (1958), 601–615.

This very informative paper concerns the hypersonic Couette flow in an externally applied magnetic field which is directed perpendicular to the walls. The configuration is the limiting case of the flow between concentric cylinders produced by the uniform motion of the exterior cylinder. The exact equations then reduce to a set of ordinary differential equations. The boundary conditions are carefully discussed and formulated. Thermodynamic equilibrium is assumed, and the pressure is taken as constant throughout the flow field. Reasonable variations of the viscosity, electrical conductivity, and Prandtl number are used to obtain numerical solutions of the equations. The effects of the magnetic field in velocity, temperature, current density, induced magnetic field, skin friction, and heat transfer are shown in numerous graphs. The results indicate that a relatively weak field produces a large reduction in skin friction, but has little effect on heat transfer. *H. Greenspan* (Cambridge, Mass.)

2967:

Bush, William B. Magneto-hydrodynamic-hypersonic flow past a blunt body. J. Aero. Sci. 25 (1958), 685–690, 728.

This paper attempts to determine the body shape and applied magnetic field that produce a given spherical shock and a specified magnetic dipole field in the free stream. Note that the applied and induced fields must superpose to form this specified free stream magnetic dipole field, although the applied field is produced by currents in the body, and the induced field arises from currents in the plasma behind the shock. The electric conductivity is assumed constant behind the shock, zero in the free stream; heat conduction and viscosity effects are neglected. The author limits himself to a discussion of the effects near the stagnation point (small polar angle), in which case the flow is approximately incompressible. Following Lighthill's analysis [J. Fluid Mech. 2 (1957), 1–32; MR 19, 352; pp. 28–31], the problem is reduced to two nonlinear ordinary differential equations with boundary conditions. For certain values of a parameter, the body shape is spherical. The separation distance between shock and body is shown to increase as the strength of the freestream magnetic field increases. No details are given concerning the nature of the applied fields. *H. Greenspan* (Cambridge, Mass.)

2968:

Woltjer, L. On hydromagnetic equilibrium. Proc. Nat. Acad. Sci. U.S.A. 44 (1958), 833–841.

Fundamental equations that must be satisfied by the stable equilibrium configurations of a given hydromagnetic system are deduced by a variational principle in which the total energy of the system is made an extremum while the integrals of the equations of motion are kept constant. It appears that, to obtain a solution that is not force-free, the fluid must have a non-vanishing velocity field. If no motions are present, the magnetic field is force-free. *C. H. Papas* (Pasadena, Calif.)

2969:

Braginskii, S. I. Transport phenomena in a completely ionized two-temperature gas. Soviet Physics JETP 33 (6) (1958), 358–369 (459–472 Ž. Eksper. Tehn. Fiz.).

L'auteur écrit les équations cinétiques (équations de Boltzmann) pour un fluide totalement ionisé; ces équations doivent être vérifiées par les deux fonctions de distribution des vitesses (fonction relative aux ions et fonction relative aux électrons). La très faible valeur de la masse des électrons par rapport à la masse des ions permet de résoudre séparément les deux équations cinétiques. La solution est obtenue par approximations; la première approximation est égale pour chacune des deux fonctions à la distribution de Maxwell; la seconde approximation, suivant la méthode de Chapman et Cowling [The mathematical theory of non-uniform gases, Cambridge, 1939; MR 1, 187], est exprimée à l'aide des polynomes de Sonine. Le flux thermique et le tenseur de viscosité correspondant à cette seconde approximation sont calculés de façon explicite. *H. Cabannes* (Marseille)

2970:

Braginskii, S. I. The behavior of a completely ionized plasma in a strong magnetic field. Soviet Physics JETP 33 (6) (1958), 494–501 (645–654 Ž. Eksper. Tehn. Fiz.).

Im ersten Teil der Arbeit werden einige scheinbar paradoxe Folgerungen besprochen, die in der Theorie eines Plasmas auftreten, wenn die Stossfrequenz $1/\tau$ viel kleiner als die vom magnetischen Felde verursachte Larmorfrequenz (Gyrationsfrequenz) ω ist. Für die Geschwindigkeiten der Elektronen v_e und der Ionen v_i werden ausserdem Formeln angegeben. Ebenfalls wird die Geschwindigkeit, mit der sich der Mittelpunkt des von einem geladenen Teilchen im Magnetfeld beschriebene Kreis entlang der magnetischen Kraftlinien bewegt, besprochen. Zuletzt werden noch Betatronwirkungen in solch einem Plasma betrachtet.

Im zweiten Teil wird die Kontraktion eines vollständig ionisierten Plasmas unter der Wirkung seines eigenen magnetischen Feldes berechnet. Es wird angenommen, dass Änderungen im Plasma so langsam verlaufen, dass man Trägheitseffekte vernachlässigen kann, weiter dass das Plasma so dicht ist, dass man für die Dichten $n_i=n_e=n$ setzen kann und dass die Ionen- und Elektronentemperaturen einander gleich sind. Die unter diesen Bedingungen bestehenden Grundgleichungen, die vom Verfasser bereits hergeleitet wurden [siehe #2969 obenstehend] werden angegeben, und danach wird zuerst der stationäre Fall betrachtet. In diesem muss das Plasma entweder an die Wände Wärme abgeben, oder wenn es sich von ihnen schon losgelöst hat, durch Strahlung Energie verlieren.

Im nichtstationären Fall wird die Lösung der Grundgleichungen sehr schwierig. Deshalb wird nur der Fall betrachtet, in dem in einem Querschnitt des Plasmas die physikalischen Grössen sich nur auf die Weise mit der Zeit ändern, dass ihre Verteilung sich ähnlich bleibt. In diesem Fall reduziert sich das Problem auf die Lösung einer Besselschen Differentialgleichung, und die Grössen H, n und E werden deshalb mit Hilfe von Besselschen Funktionen imaginären Argumentes ausgedrückt.

Die Berechnungen des Verfassers rühren noch vom Jahre 1952 her und seine Resultate stimmen grösstenteils

mit denen von A. Schlüter [Z. Naturf. **5a** (1950), 72–78] und M. Kruskal und M. Schwarzschild [Proc. Roy. Soc. London Ser. A **223** (1954), 348–360; MR **15**, 914] überein.}

T. Neugebauer (Budapest)

2971:

Crupi, Giovanni. Su una nuova equazione delle onde piane magneto-idrodinamiche propagantisi in una generica direzione ed una sua applicazione. Boll. Un. Mat. Ital. (3) **13** (1958), 173–178.

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 2855, 2891, 2895, 2969, 2970, 3027, 3030, 3040.

2972:

Buchdahl, H. A. Optical aberration coefficients. II. The tertiary intrinsic coefficients. J. Opt. Soc. Amer. **48** (1958), 563–567.

The thoroughness and painstaking care which the author has applied to the development of the algebraic theory of aberrations of symmetrical optical systems [Optical aberration coefficients, Oxford Univ. Press, London, 1954; MR **19**, 354; and the first paper of the present series, J. Opt. Soc. Amer. **46** (1956), 941–943; MR **19**, 355] is extended here to obtain relatively simple (and computable) forms for the monochromatic tertiary, i.e., seventh order, intrinsic aberration coefficients of a system of spherical surfaces. The algebraic steps leading to an explicit form of higher-order coefficients are elementary, but very tedious, and lead to an almost unmanageable jungle of terms unless a carefully contrived notation is used. In the previous paper the author has shown that the introduction of a Seidel form simplified the calculation of tertiary spherical aberration so that only 19 entries per surface, opposed to 121 entries previously, were required. It is shown here that, given the quantities obtained in computing the secondary coefficients, the computation of all 10 tertiary intrinsic coefficients requires only 23 entries per surface. A further paper incorporating the present results in a complete computing scheme is promised.

G. L. Walker (Southbridge, Mass.)

2973:

Tai, C. T. The electromagnetic theory of the spherical Luneberg lens. Appl. Sci. Res. B. **7** (1958), 113–130.

The paper contains a mathematical treatment of the so-called Luneberg lens which has the geometrical optics property of transforming the rays issued from a point source, placed on the boundary of a sphere, into a parallel beam after refraction by the solid sphere with an index of refraction of the form $n=(2-r^2/a^2)^{\frac{1}{2}}$, a being the radius of the sphere. Starting with Maxwell equations and the divergence relations, the author has constructed the vector wave functions from the solution of the scalar wave equation for the case of a stratified medium in the radial direction. The field quantities E, H are expressed in series of these functions. The radial differential equation of the scalar wave equation, for transverse electric modes, reduces to an ordinary equation of the confluent hyper geometric type and, for the magnetic modes, to an equation, possessing two finite singularities, which resembles a confluent type. The rest of the paper deals with a discussion of this equation. A solution of this equation in the neighborhood of the origin is given in the form of an ordinary power series. The incident and scattered fields, due to a source in the form of a horizontal electric dipole located outside the sphere, are expressed in series of vector wave functions. The coefficients of these series are determined from the boundary conditions on the sphere. The far zone field and the scattered field resulting from an incident plane wave are also exhibited.

N. Chako (Flushing, N.Y.)

2974:

Miyamoto, Kenrō. Comparison between wave optics and geometrical optics using Fourier analysis. II. Astigmatism, coma, spherical aberration. J. Opt. Soc. Amer. **48** (1958), 567–575.

In part I of this investigation [same J. **48** (1958), 57–63; MR **19**, 1009], the relation between wave optics and geometrical optics was studied, with the help of Fourier analysis. In the present paper the results are used to compare the geometric optical response function R_g with the wave optical response function R_w for the cases of spherical aberration, coma and astigmatism. Conditions are found under which $R_g \sim R_w$.

E. Wolf (Manchester)

2975:

✱Румер, Ю. Б. Исследования по 5-оптике. [Rumer, Yu. B. Studies in 5-dimensional optics.] Zap.-Sibir. Filial Akad. Nauk SSSR. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 152 pp. 3.80 rubles.

This monograph is a revised version of various papers published by the author in the last ten years on the subject of a unified five dimensional, generally covariant quantum theory of fields and elementary particles. It starts with the optical mechanical analogy in a five dimensional formulation. It then treats a five dimensional geometrical and wave optics, first in a Minkowski space, and then generalises the results to a five dimensional Riemann space.

P. Roman (Manchester)

2976:

Battig, A. Incidencia de una onda electromagnetica plana sobre un gas de electrones en movimiento uniforme dentro de un medio dielectrico. Univ. Nac. Tucumán. Rev. Ser. A. **11** (1957), 110–119.

2977:

Iwata, Giiti. Perfect imaging dynamical systems. Progr. Theoret. Phys. **19** (1958), 375–388.

Perfect imaging systems in optics as exemplified by Maxwell's fish eye, Stettler's generalized fish eyes, the Luneburg lens, etc., have recently attracted increasing attention because of their analogues in particle optics. The present paper classifies perfect imaging dynamical systems from the standpoint of particle optics and studies their properties. Previous results of the author [Progr. Theoret. Phys. **8** (1952), 183–192; **9** (1953), 97–107; MR **14**, 603, 1134] on a system with a Lagrangian not explicitly involving the time are extended to the time dependent case.

The perfect imaging systems of classical optics are conformal mappings. The author asserts that translations and rotations are not within the scope of optical systems and so unaccountably excludes mirrors. In any case, he gives examples, among others, of systems realizing these transformations in particle optics by virtue of magnetic fields.

G. L. Walker (Southbridge, Mass.)

2978:

Lüst, R.; und Schlüter, A. Die Bewegung geladener Teilchen in rotations-symmetrischen Magnetfeldern. Z. Naturf. **12a** (1957), 841–843.

Es wird die Frage untersucht, durch welche zylinder-

symmetrischen Magnetfelder es möglich ist, Teilchen innerhalb eines geschlossenen Volumens festzuhalten. Ein zylindersymmetrisches Magnetfeld wird in ein toroidales und in meridionales Feld zerlegt. Beide Anteile werden durch eine skalare Funktion beschrieben. Die Verfasser gehen von den Bewegungsgleichungen eines geladenen Teilchens in einem Magnetfeld aus und ermitteln durch direkte Integration die toroidale Geschwindigkeit, welche durch die meridionale Magnetfeldfunktion bestimmt wird. Für die meridionale Beschleunigung ergibt sich eine Gleichung aus welcher hervorgeht, dass die toroidale Komponente des Magnetfeldes nur auf die Bewegung in der Meridianebene einen Einfluss hat. Die erhaltenen Gleichungen werden auf die Bewegung geladener Teilchen in einem Dipolfeld angewandt. Es lassen sich Gebiete bestimmen, welche von den Teilchen mit bestimmter Geschwindigkeit und mit bestimmten Drehimpuls nicht verlassen werden können. Schliesslich ergibt sich dass jedes magnetische Feld mit einem meridionalen Anteil, dessen Feldlinien geschlossen sind, Teilchen mit nicht zu hoher Energie dauernd in einem endlichen Volumen festhalten kann. Für ein Plasma mit hoher Temperatur ist die obige makroskopische Beschreibung des Gleichgewichtes nicht hinreichend, wenn die freien Weglängen gross gegen die linearen Dimensionen des Volumens werden, in dem das Plasma festgehalten werden soll.

M. J. O. Strutt (Zürich)

2979:

Horváth, J. I. Eine Axiomatisierung der Maxwellschen Theorie des elektromagnetischen Feldes. Acta Phys. Acad. Sci. Hungar. **8** (1958), 399–418. (Russian summary)

Für die beabsichtigte Axiomatisierung der Maxwellschen Theorie des elektromagnetischen Feldes nimmt sich Verfasser G. Hamel's Axiomatisierung der Mechanik zum Vorbild. Dementsprechend wird von den Axiomen einer physikalischen Theorie verlangt: (a) Vollständigkeit; (b) Widerspruchslosigkeit; (c) Unabhängigkeit; (d) Realisierbarkeit. Die vierte Forderung bedeutet Übereinstimmung mit der Erfahrung.

Nach einer wohlgelungenen Darstellung der experimentellen Grundlagen der Maxwellschen Theorie gliedert Verfasser die Axiome der Maxwellschen Theorie in fünf Gruppen: Erzeugungsaxiome (E), Existenzaxiome (Ex), Zustandsaxiome (Z), Verknüpfungsaxiome (V) und Materialaxiome (M). Sie lauten: E.I. Das elektromagnetische Feld wird durch die elektrischen Ladungen und durch die elektrischen Ströme erzeugt; E.II.A. Die elektrischen Ladungen sind a priori Gegebenheiten; E.II.B. Es gibt zwei und nur zwei verschiedene Arten der elektrischen Ladungen; Ex. Das Vorhandensein des elektromagnetischen Feldes wird durch die mechanische Messung seiner ponderomotorischen Kraftwirkung und seiner Energie erkannt; Z.I. Der Zustand des elektrischen Feldes wird durch seine Intensität und durch seinen Erregungszustand charakterisiert; Z.I.A. Die Intensität des elektrischen Feldes wird durch die Einführung des Vektors der elektrischen Feldstärke beschrieben; Z.I.B. Der Erregungszustand des Feldes wird durch den Erregungsvektor gekennzeichnet; M.I. Die Körper, die in der Natur vorhanden sind, werden in elektrische Leiter und Isolatoren eingeteilt; M.II.A. Die in einem Leiter durch den Querschnitt des Leiters in der Zeiteinheit hindurchtretende Ladungsmenge wird der Leitungsstrom genannt; M.II.B. Unter Leitungsstromdichte wird die in der Zeiteinheit durch die zur Stromrichtung senkrechte Flächeneinheit gehende Summe der Ladungsmengen verstanden; E.II.C. Die Zeitableitung des Erregungsvektors des elektrischen Feldes wird die Erregungsstromdichte genannt; M.II. D. Die Dichte des elektrischen Gesamtstromes läßt sich aus der Leitungsstromdichte und aus der Erregungsstromdichte additiv zusammensetzen; Z.II. Der Zustand des magnetischen Feldes wird durch seine Intensität und durch seinen Erregungszustand charakterisiert; Z.II.A. Die Intensität des magnetischen Feldes wird durch den magnetischen Induktionsvektor beschrieben; Z.II.B. Der Induktionsfluß durch eine beliebige geschlossene Fläche verschwindet; Z.II.C. Der Erregungszustand des magnetischen Feldes läßt sich durch seine Magnetisierungsfähigkeit und durch die das Feld erzeugende elektrische Gesamtstromstärke bestimmen. V.I. Die Veränderungen des elektrischen und magnetischen Feldes werden miteinander durch eine unmittelbare und unaufhebbare Wechselwirkung verknüpft; V.II. Längs einer beliebigen geschlossenen Kurve wird infolge der zeitlichen Veränderung des ganzen Induktionsflusses durch die von der Kurve begrenzte Fläche eine bestimmte elektrische Ringspannung erzeugt; M.II. Der materiefreie Raum besitzt keine elektrische und magnetische Eigenschaft; M.III.A. Die elektrische Eigenschaft der Isolatoren wird durch die elektrische Polarisation charakterisiert; M.III.B. Die magnetische Eigenschaft der Leiter wird durch ihre Magnetisierung charakterisiert; M.IV. Die elektrische Leitungsstromdichte in einem Leiter hängt linear von der elektrischen Gesamtfeldstärke ab.

Aus dem Axiomensystem wird insbesondere das Erhaltungsgesetz der elektrischen Ladung abgeleitet. Die weitere Diskussion behandelt vornehmlich die elektro- und magnetostatische Theorie, das Biot-Savartsche Kraftgesetz der stationären Ströme und die Existenz elektromagnetischer Wellen. Die Grenzen der Gültigkeit der Maxwellschen Theorie sind erreicht, wenn man die atomistische Struktur von Elektrizität und Materie in Betracht ziehen will.

M. Pinl (Cologne)

2980:

Boudouris, G. Une nouvelle solution du problème de propagation au-dessus d'une terre plane. Nuovo Cimento (10) **5** (1957), supplemento, 71–91.

This article deals with the famous Sommerfeld problem of the propagation of electric waves from a vertical electric dipole source in the presence of a plane semiconducting homogeneous medium of infinite extent. The source is located in air (vacuum) at some distance from the plane boundary separating the two media. The author starts with the assumption that the dielectric constant of the medium is very large in comparison with that of air, and limits his analysis to the case in which the observation points and the source lie on a plane normal to the boundary of separation (meridional plane), i.e., the x-z plane. By expressing the field in a geometrical wave form with the presence of a slow varying damping factor, the electric field in the medium is determined by successive approximations of the solution of the wave equation which, in this case, reduces to a successive system of equations (non-homogeneous) in the single variable z. By following exactly the method given in the book of Y. A. Alpert, V. L. Ginsberg and E. L. Feinberg ["The propagation of radio waves." Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953; MR 19, 211; chapters IV and VII], the author has calculated the total Hertzian vector up to second order approximation. These results are in agreement with those derived by other methods by Weyl and van der Pol. The formulas derived

here are also valid for moderately large angles of incidence. Finally, it is shown that the expressions representing the reflected field lead to identical formulas given previously by Norton. The author emphasizes that by taking the approximations into account at the beginning of the analysis, rather than by first treating the problem in all of its generality and afterwards making the approximations, the calculations are simplified greatly, and the final result remains the same.

N. Chako (Flushing, N.Y.)

2981:

Banfi, Carlo. Propagazione di onde elettromagnetiche piane in un conduttore unidirezionale con direzione di conduttività, variabile. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 306–310.

2982:

Wait, James R. Transmission and reflection of electromagnetic waves in the presence of stratified media. J. Res. Nat. Bur. Standards **61** (1958), 205–232.

The formal solution for the reflection coefficient for plane electromagnetic waves of a stratified medium composed of M homogeneous layers is obtained and specialized to some simple cases. The excitation of such a medium by line and point sources is then formulated in terms of this reflection coefficient and results are calculated for several well-known special cases, such as a homogeneous half-space, a thin dielectric layer, and an impedance boundary. Although no new solutions are obtained, the paper serves as a review with a unified approach to several important problems, and some numerical calculations are presented.

E. T. Kornhauser (Providence, R.I.)

2983:

Silverman, Richard A. Scattering of plane waves by locally homogeneous dielectric noise. Proc. Cambridge Philos. Soc. **54** (1958), 530–537.

In the usual approach to the problem of weak scattering of a plane wave (acoustic or electromagnetic) by random fluctuations of refractive index, the fluctuations are assumed to exist only within a finite volume, and to be statistically homogeneous in space there. In the approach put forward in this paper, inhomogeneity is permitted, but the correlation between refractive index fluctuations at any two points $\mathbf{r}$ and $\mathbf{r}'$ is assumed to have the form $F(\frac{1}{2}\mathbf{r}+\frac{1}{2}\mathbf{r}')C(\mathbf{r}-\mathbf{r}')$. The scattered power and the space correlations of the scattered wave are calculated using both approaches, and the second approach is seen to result in a neater, more exact and more general treatment.

I. Howells (Cambridge, England)

2984:

✶Rubinowicz, A. Die Beugungswelle in der Kirchhoffschen Theorie der Beugung. Polnische Akademie der Wissenschaften. Państwowe Wydawnictwo Naukowe, Warsaw, 1957. 305 pp. (4 plates)

In looking over the scientific literature of the past fifteen years, one notices a revival of interest in the theory of diffration of light and electromagnetic waves and its application to problems of practical significance which one could have hardly foreseen half a century ago. These new contributions have already found their place in several excellent books and monographs. Consequently, one wonders whether the appearance of a treatise on the classical theory of diffraction is perhaps somewhat superfluous. The answer can easily be found by glancing through this book. Professor Rubinowicz's treatise is not only welcomed as an important contribution to the present literature on diffraction, but it is also timely in two important respects. In the first place, there exists, that the reviewer is aware of, no treatise which covers in such a comprehensive manner the classical theory of diffraction (Baker and Copson, The mathematical theory of Huygens' principle [Oxford Univ. Press, New York, 1939; MR **1**, 315] is an exception) and, secondly, the physical aspects and interpretations of the mathematical theory are brought forth in a clear and balanced form rarely found in books of this kind.

This book, which is dedicated to the memory of the maestro of theoretical physics, the late Professor Arnold Sommerfeld, may be divided in three parts: (a) the fundamental principles and interpretations of diffraction phenomena, as derived from the ideas of Huygens, Young and Fresnel, the mathematical foundations of the theory as formulated by Kirchhoff, and its modification, principally by Rayleigh, Sommerfeld and Kottler; (b) the physical aspects of the diffracted wave, toward the elucidation of which the author has been one of the principal contributors; and (c) a critical discussion and analysis of the diffracted wave phenomena arising in different specific problems and the comparison of the results so obtained with those derived from the exact theory of diffraction.

In the first two chapters one finds a clear and detailed exposition, which is rarely seen even in modern books and monographs, of diffraction phenomena, based on the principles of Huygens, Young and Fresnel, and on Kirchhoff's formulation of Huygens' principle and its various modifications due to Rayleigh, Sommerfeld and Kottler. This is followed by a detailed account of the physical basis of Kirchhoff's theory, the modification of this theory by Sommerfeld in terms of Green's functions, the discontinuous boundary value problem of the wave equation and a brief analysis of two-dimensional diffraction. In the third chapter the author discusses the method by which Kirchhoff's integral is separated into a geometrical wave and a diffracted wave for the case of spherical waves. The procedures first given by Maggi and later derived by another method by the author are described in detail. The physical interpretation of the solution and the character of the diffracted wave across shadow boundareis and focal planes are analyzed. Chapter four contains an extensive treatment of diffraction by a half-plane, on the basis of Kirchhoff's theory, and also the Sommerfeld solution. Here, the author shows that the discontinuity appearing in the geometrical wave across the shadow boundary is cancelled by the diffracted wave, resulting in a continuous solution across the boundary. The results derived from Kirchhoff's integral are found to be equivalent to those obtained from Sommerfeld's solution for points near the shadow boundary. Approximate expressions for the wave function are given for points away from the shadow boundary by the application of the method of stationary phase. The range of validity of this method is discussed in detail, including cases where the observation points are near or on focal planes and in the neighborhood of the geometrical shadow. The following chapter contains an account of the case of geometrical optics approximation, by applying the method of stationary phase to the integral representing the diffracted wave. This is followed by a critical analysis of the validity of the results deduced by the stationary phase method and the physical interpretations of these approximations, especially in the neighborhood of the shadow. The author also discusses the so-called active regions of boundary

stationary points for convergent spherical waves, as well as divergent waves, including the interpretation of the experimental results in the light of the above analysis. The final section contains a clear treatment of Fraunhofer diffraction phenomena.

Chapter six deals primarily with Kirchhoff's formulation of diffraction (diffracted wave) for electromagnetic waves and electron waves, as well as the rigorous solution of electromagnetic diffraction. In the final chapter the author is mainly concerned with the general consideraations of the problem of reflection and diffraction of waves, Huygens' wave construction (envelope principle) and the uniqueness of the initial value problem for Maxwell equations, and the propagation of discontinuities of the electromagnetic fields. The concluding pages contain an extensive bibliography and a subject index.

The book can be warmly recommended to all readers, specialists and non-specialists alike, for its clear and comprehensive treatment of the classical theory of diffraction; to teachers of optics and, particularly, to students of optics who wish to acquire a better understanding of one of the most beautiful topics in the field of optics and electromagnetic theory; and to applied mathematicians interested in wave propagation in general. The reviewer is of the opinion that a new English edition incorporating recent advances would enhance considerably the value of the book, and also would make it available to a large circle of readers.

N. Chako (Flushing, N.Y.)

2985:

Pitteway, M. L. V. The reflexion of radio waves from a stratified ionosphere modified by weak irregularities. Proc. Roy. Soc. London. Ser. A. **246** (1958), 556–569.

A method is developed for calculating the field scattered from weak irregularities in a stratified ionosphere, when the Earth's magnetic field is neglected. An integral of the equations satisfied by a small perturbation field is obtained in terms of the unperturbed field and an arbitrary field satisfying the unperturbed Maxwell equations.

The integral formula is applied to the region, between two horizontal planes, which contains the ionosphere, for the case of a plane wave incident at any angle. By choosing the arbitrary field as a plane wave incident below the ionosphere, a formula for the scattered component in the opposite direction is obtained.

The result is used to examine the suggestions that irregularities might cause exceptionally strong scattering (a) near the level of reflection as given by ray theory, and (b) by a "plasma resonance" process, when the electric vector is directed across the irregularities at the level where the refractive index is zero. It is found that, in case (a), there is no pronounced enhancement of the scattering and, in case (b), the normal electron collision frequencies in the ionosphere are such as to damp out any enhancement due to plasma resonance.

K. C. Westfold (Pasadena, Calif.)

2986:

Millar, R. F. Diffraction by a wide slit and complementary strip. I, II. Proc. Cambridge Philos. Soc. **54** (1958), 479–511.

A plane monochromatic electromagnetic wave (wave number k) falls onto a thin perfectly conducting plane screen consisting of two half-planes separated by an infinite slit of width h. The direction of incidence is perpendicular to the edges of the slit and makes an angle α with the normal to the screen. In part I [II] the electric [magnetic] vector is parallel to the edges. Integral equations and differential-integral equations, respectively, are set up for the unknown currents in the two half-planes. In both cases these equations are transformed into a pair of simultaneous integral equations, which are then solved by a process of successive substitutions. The solution obtained (and essentially due to Schwarzschild) is asymptotically evaluated for large values of hk. Asymptotic expansions are also obtained for the tangential electric field in the slit and for the far-field behind the slit. Special attention is paid to asymptotic expansion of the transmission coefficient. Application of Babinet's principle solves the analogous problems for the strip. The author's results supersede all previous results on the transmission coefficient, in that he obtains asymptotic expansions holding for all values of α. Comparison with exact numerical results available from the literature shows that the asymptotic expansions, so far as the first few terms are concerned, are numerically useful if $hk \gtrsim 4$.

C. J. Bouwkamp (Eindhoven)

2987:

Reza, F. M. A multiplication theorem for positive real functions. Proc. Amer. Math. Soc. **9** (1958), 496–499.

The principal result of this paper is that if $z_1, z_2, \cdots, z_n$ are positive real functions then

$$\frac{\prod_1^n (1+z_i)-\prod_1^n (1-z_i)}{\prod_1^n (1+z_i)+\prod_1^n (1-z_i)}$$

is a positive real function. This is an immediate generalization involving no essential change in proof of theorem 1(i) given by Fialkow and Gerst [J. Math. Phys. **34** (1955), 160–168; MR **17**, 435; p. 161]. This latter theorem, in turn, is an easy consequence of the multiplication theorem for unimodular bounded functions. The problem of the construction of simpler positive real functions from a given one solved by Richards [Duke Math. J. **14** (1947), 777–786; MR **9**, 181] and by Fialkow and Gerst [theorem 1(ii), loc. cit.] is considered again by the present author, but no satisfactory solution is given.

A. Fialkow (Brooklyn, N.Y.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 2534, 2600, 2950, 2951, 2952, 2956, 2957, 2958, 2959, 3019.

2988:

Hvingiya, L. V. On a solution of the differential equation of heat conduction for bodies of complicated shape. Soobšč. Akad. Nauk Gruzin. SSR **20** (1958), 257–264. (Russian)

The author states that the solution of the heat equation, with given constant initial temperature and Newtonian cooling on the exterior surface, for certain hollow bodies of complicated shape can be reduced to the same problem for a cylindrical shell of finite thickness. In the equivalent problem cooling occurs only on the lower and exterior lateral surface. The solution is obtained as the product of solutions of the heat equation for an infinite plate and an infinite cylindrical shell with appropriately chosen boundary conditions.

D. G. Aronson (Flushing, N.Y.)

2989:

Dorfman, L. A. Influence of the radial temperature gradient on the heat transfer from a rotating disk. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 12, 64–66. (Russian)

The author solves the heat flow equation for a rotating

disc under boundary conditions imposed by a radial quadratic temperature distribution. The solution takes into account energy dissipation in the boundary layer, and indicates the influence of the radial temperature gradient. By an example, it is shown that the heat transfer is multiplied by more than one and one-half when there is a quadratic temperature gradient along the radius.

M. D. Friedman (Needham Heights, Mass.)

QUANTUM MECHANICS

See also 2278, 2503, 2608, 2892, 2893, 2894, 2895, 2896, 2897, 2933, 2975.

2990:

Gerjuoy, E. Time-independent nonrelativistic collision theory. Ann. Physics **5** (1958), 58–93.

Cross sections for the scattering of a single particle by a potential are often computed, not from transition probabilities, but from the asymptotic form of the wave function at infinity. Such calculations are extended in the present paper to arbitrarily complicated rearrangement collisions describable by the time-independent non-relativistic Schroedinger equation. This is accomplished by means of the introduction of a generalized probability current operator for many particle systems. The formulae for the reaction rates are identical with those obtained from the conventional formalism. It follows immediately from the asymptotic behavior of the Green's function that the matrix elements $\Psi_f^{(-)*}V_i\psi_i=\psi_f^*V_f\Psi^{(+)}$ yield the scattering amplitude from state i to state f, and that $\Psi_f^{(-)}$ is a solution with incoming scattered waves. Here, Ψ and ψ satisfy the Schroedinger equations $(H-E)\Psi=0$ and $(H_i-E)\psi=0$, respectively, where H is the total Hamiltonian $H=H_i+V_i$, V_i being the potential. One also gets automatically, without projecting the solutions on plane wave states, that coincidences are observed in counters placed at large distances only when these distances are in the ratios of the classical velocities. The investigation is carried out in the laboratory system, since it appears that the elimination of the center-of-mass coordinates yields no simplification. The appendix discusses the asymptotic behavior of the Green's function and of the various integrals occurring in the calculations, as well as the mathematical properties of the current operator.

M. J. Moravcsik (Livermore, Calif.)

2991:

Fock, V. On the Schrödinger equation of the helium atom. I, II. Norske Vid. Selsk. Forh., Trondheim, **31** (1958), no. 22, 7 pp.; no. 23, 8 pp.

English translation of the original Russian paper [Izv. Akad. Nauk SSSR. Ser. Fiz. **18** (1954), 161–172; MR **17**, 334].

2992:

Budini, P. On the regular representation of the Lorenz group. Univ. Nac. Tucumán. Rev. Ser. A. **11** (1957), 84–93.

The paper begins with the ostensibly false statement that every irreducible representation of the homogeneous Lorentz group is contained in the regular representation. However, it turns out that the author does not impose the requirement of square integrability in defining the regular representation and therefore the statement is correct if suitably interpreted. In fact, the main point of the paper is to prove that eigenfunctions of the invariant operators $F=\frac{1}{4}(M_{\mu\nu}M^{\mu\nu})$ and $G=\frac{1}{4}\varepsilon_{k\lambda\mu\nu}M^{k\lambda}M^{\mu\nu}$ are not square integrable for $G=0$ and $0>F>-\frac{1}{4}$. This result is well known [I. M. Gelfand and M. A. Naimark, Izv. Akad. Nauk. SSSR Ser. Mat. **11** (1947), 411–504; MR **9**, 495]. It is stated that this shows "that the corresponding states are not realizable in physics". The representation actually studied here is obtained from the transformations of functions defined on the manifold with coordinates $x_1, \cdots, x_4, y_1, \cdots, y_4$ which is defined by $\sum_{j=1}^4 x_j^2-\sum_{j=1}^4 y_j^2=1$; $\sum_{j=1}^4 x_jy_j=0$.

A. S. Wightman (Princeton, N.J.)

2993:

Polkinghorne, J. C. Generalized retarded products. Proc. Roy. Soc. London. Ser. A. **247** (1958), 557–561.

In recent years the concept of a retarded commutator has become more and more popular in the theory of quantized fields. As far as the reviewer knows, a retarded commutator in a modern context was first used by H. Umezawa and S. Kamefuchi in 1951 [Progr. Theoret. Phys. **6** (1951), 543–558; MR **13**, 713], who studied the particular case of the retarded commutator of two operators in connection with the problem of vacuum polarization in quantum electrodynamics. Afterwards, several authors have, more or less independently, rediscovered the usefulness of the retarded commutator and introduced suitable generalizations for more than two operators. In what can perhaps nowadays be considered the standard form, the space time point of one of the operators involved is singled out and the retarded commutator vanishes unless all other space time points are earlier than the distinguished point. In the paper by Polkinghorne reviewed here, this generalization is carried one step further and a generalized retarded commutator is introduced for which the space time points involved are divided in two sets. Every point in the first set has to be later than every point in the second set for this generalized commutator to be different from zero, while there is no such strong restriction between the points within the same set. When the first set contains only one point, one has the conventional retarded commutator, while one gets the conventional advanced commutator when the second set contains only one point. It is shown that any matrix element of the S-matrix can be expressed in terms of vacuum expectation values of these generalized retarded commutators in a way analogous to the formula given already by Umezawa and Kamefuchi for their special problem and usually referred to as the reduction formula technique of Lehmann, Symanzik and Zimmermann [Nuovo Cimento **6** (1957), 319–333; MR **19**, 1133].

G. Källén (Lund)

2994:

Pécheux, Michel. Etats liés et pathologiques dans le modèle d'interaction méson-nucléon de Bosco et Stroffolini. Cahiers de Phys. **91** (1958), 110–128.

The author considers a very simplified model for the interaction of π-mesons and nucleons. In this model, which was originally introduced by Bosco and Stroffolini [Nuovo Cimento (10) **2** (1955), 433–442], one neglects all terms in the interaction Hamiltonian except those giving rise to transitions where a π-meson is destroyed and a nucleon anti-nucleon pair is created. This model has a certain formal similarity to the Lee model [T. D. Lee, Phys. Rev. (2) **95** (1954), 1329–1334; MR **16**, 317], and the author shows that one can obtain an explicit solution for some of the simplest states involved. Because of the

spin and isotopic spin variables of the nucleons, the actual computational work is somewhat more involved than in the Lee model, but the general techniques are the same. It also turns out that the results are rather similar for the two cases, as the author shows that one has exactly the same difficulties with anomalous states and negative probabilities in the Bosco-Stroffolini model as in the Lee model [G. Källén and W. Pauli, Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), no. 7; MR **17**, 927].

G. Källén (Lund)

2995:

Greenberg, O. W.; and Schweber, S. S. Clothed particle operators in simple models of quantum field theory. Nuovo Cimento (10) **8** (1958), 378–406.

Two papers of Van Hove [Physica **21** (1955), 901–923; **22** (1956), 343–354], which analyzed the relation of bare particles to their associated cloud of mesons created by the interaction hamiltonian, were the starting point of the present investigation. Bare particles which are eigenstates of the free-field hamiltonian are physically unreal, so the authors propose to formulate field theory in terms of clothed particles which are eigenstates of the total hamiltonian. Self-energy and cloud effects associated with one-particle states are automatically eliminated, and the simplest vertex type occurring in the expansion of the S-matrix corresponds to fermion-boson scattering. The programme is carried out explicitly for (1) a neutral scalar field interacting with fermions, (2) the Lee model [Phys. Rev. **95** (1954), 1329–1334; MR **16**, 317] which involves a boson field interacting with two fermion fields, (3) the simplest Ruijgrok-Van Hove model [Physica **22** (1956), 880–886]. The hamiltonian is given, in each case, in terms of the clothed particle states. A unitary operator which transforms bare particles into clothed particles is obtained, exactly for (1) and (2) and approximately for (3).

A. J. Coleman (Toronto, Ont.)

2996:

Karplus, Robert; Sommerfield, Charles M.; and Wichmann, Eyvind H. Spectral representations in perturbation theory. I. Vertex function. Phys. Rev. (2) **111** (1958), 1187–1190.

The analytical properties of the vertex function due to the interaction of three scalar meson fields coupled to three scalar intermediate fields is investigated in lowest order perturbation theory. The function depends only on the invariant momentum transfer q^2 and is found to be regular in a plane cut along the negative real axis, $q^2<-\mu^2<0$. A dispersion relation ("spectral representation") therefore exists. The dependence of μ on the masses of the six fields is studied.

F. Rohrlich (Baltimore, Md.)

2997:

Eliezer, Jayaratnam, C. A consistency condition for electron wave functions. Proc. Cambridge Philos. Soc. **54** (1958), 247–250.

The Dirac equation for an electron in an electromagnetic field is considered from the viewpoint of expressing this field as a function of the spinor amplitudes (which is the converse of the usual problem). Such a relation is found, giving, for example, the vector potential **A** in terms of bilinear forms in ψ and of the scalar potential; it is clear from gauge considerations that not all four components A_μ could be independently determined by ψ. This result is analogous to one in the coupled Maxwell-Einstein problem (a very different domain), in which it is possible to read from the metric variables the electromagnetic fields inducing the curvature [Misner and Wheeler, Ann. Physics **2** (1957), 525–603; MR **19**, 1237]. A consistency condition which must be satisfied by the electron amplitude is also noted. All of the results of the paper deal with c-number amplitudes.

S. Deser (Waltham, Mass.)

2998:

Norton, Richard; and Klein, Abraham. Significance of the redundant solutions of the Low-Wick equation. Phys. Rev. (2) **109** (1958), 991–995.

Using a special example, it is demonstrated that the Low equation [Phys. Rev. (2) **97** (1955), 1392–1398] does not uniquely determine the scattering amplitude, but has an infinite number of solutions. The example consists of a scalar meson field in dipole interaction with N harmonic oscillators. Since the total Hamiltonian is quadratic in all variables, the solution amounts to the classical problem of (an infinite set of) coupled oscillators. This can be treated exactly, giving rise to N resonance frequencies. On the other hand, the Low equation for this example can be constructed. It turns out that N does not occur in it, which already proves the point. In addition, the general solution of the Low equation is obtained by means of the technique of singular integral equations. This general solution can always be realized by taking a suitable combination of harmonic oscillators. {The state of affairs is analogous to the well-known case of propagation of light in a medium. The Kramers-Kronig relations do not determine the dispersion formula uniquely, but the general solution has the form of a Lorentz dispersion formula, and can therefore be realized by harmonically bound Lorentz-electrons.}

N. G. van Kampen (Washington, D.C.)

2999:

Ekstein, H.; Swihart, J.; and Tanaka, K. Representationless formalism in the field theory of fixed nucleons. Phys. Rev. (2) **109** (1958), 557–566.

Scalar and pseudoscalar charge-symmetric meson theories with fixed nucleons are studied in what the authors call the representationless formalism, which is the formalism obtained when all quantities of interest are expressed in terms of matrix elements between physical (i.e. dressed) nucleon states. The results are expressed in terms of a renormalized coupling constant, either as a power series, or as a quotient of two power series. As an application, the magnetic moment of the nucleon is calculated in the pseudoscalar theory with pseudovector coupling.

L. Van Hove (Utrecht)

3000:

Ivanov, T. F. Asymptotic solution of Thomas-Fermi equation. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 20–21. (Russian)

3001:

Biswas, S. N. General solution of the Bethe-Salpeter equation in instantaneous interaction approximation. Progr. Theoret. Phys. **19** (1958), 725–739.

The author examines the Bethe-Salpeter equation for the two nucleon problem. The ladder approximation is made with an instantaneous interaction. Thus, the interaction operator is assumed to be a function of the relative spatial coordinate times a Dirac delta function in the relative time. This simple time dependence allows the author to obtain rigorous radial equations in the spin-singlet and spin-triplet states. A non-relativistic reduction is then made and effective potentials are obtained. In both spin states, the potential becomes singular at a

finite distance from the origin. The author interprets this as a "hard core" in his potential. {The reviewer tends to feel that it is difficult to make statements about the interior part of the nuclear potential using such a simple multiplicative instantaneous potential.} A spin-orbit potential is also obtained in the triplet state. The deuteron parameters are examined and a reasonable fit of the data seems possible. The low energy singlet parameters are not compared with experiment.

R. Arnowitt (Syracuse, N.Y.)

3002:

Bell, J. S. A variational method in field theory. Proc. Roy. Soc. London. Ser. A. **242** (1957), 122–128.

If G is the complete one-nucleon propagator, T the operator $M_0+\gamma_\mu(\partial/\partial x_\mu)+ig\tau\gamma_5\phi(x)$ of a nucleon in a pseudoscalar field, and $S(\phi)$ the functional $e^{iI(\phi)}/\int e^{iI}d\phi$ then the expression $G^{-1}=\int US^{-1}TU\,d\phi$ is stationary for variations of U, U subject to $\int U\,d\phi=\int \bar{U}\,d\phi=1$. Skyrme's variational principle [Proc. Roy. Soc. London Ser. A **231** (1955), 321–335; MR **17**, 220] is a functional Fourier transform of this. The equations of variation are $TU=UT=SG^{-1}=G^{-1}S$, and these may be solved by perturbation methods, or by using trial functional forms with disposable parameters. *C. A. Hurst* (Adelaide)

3003:

Bell, J. S.; and Skyrme, T. H. R. The anomalous moments of nucleons. Proc. Roy. Soc. London. Ser. A. **242** (1957), 129–142.

The variational method described in a previous paper [#3002 above] is applied to the calculation of the magnetic moments of nucleons and the neutron-electron interaction in the symmetric *PS-PS* theory. Vacuum polarization is ignored and a very simple trial function employed. A rather rough fit to the observed values is obtained. (from the author's summary)

C. A. Hurst (Adelaide)

3004:

Kastler, Daniel. Le domaine de localisation d'une certaine classe d'états de champ. C. R. Acad. Sci. Paris **245** (1957), 2021–2023.

In a discussion of the foundations of collision theory, R. Haag has introduced the notion of localization in space time for certain states in quantum field theory [Colloque sur les problèmes mathématiques de la théorie quantique des champs, Lille, 1957]. A state of the form $P(A(f_1)\cdots A(f_n))\Psi_0$ is said to be localized in any region of space time which includes all the supports of the f_j. (Here, $A(f_j)=\int d^4x f_j(x)A(x)$, where $f_j\in\mathcal{D}$, the set of all infinitely differentiable functions of compact support on space-time; $A(x)$ is a neutral scalar field; $P(A(f_1)\cdots A(f_n))$ is a polynomial in the indicated arguments and Ψ_0 is the physical vacuum state.) In the present note, the author proposes an alternative definition which associates to each state Ψ of the form just described, a unique set G_Ψ of space time points, called the domain of localization of Ψ. The work is carried out for a free neutral scalar field and the author poses as an unsolved problem the extension to interacting fields. G_Ψ is defined as the union of the supports of functions f belonging to a certain subset K_Ψ of $\mathcal{D}$. K_Ψ is defined as

$$\{f\in\mathcal{D};\ A^{(-)}(f)\Psi=0\}^\perp.$$

Here $\perp$ denotes the orthogonal complement in the scalar product $(f,g)=\pi\int \bar{f}(p)\tilde{g}(p)d^3p/\sqrt{(\mathbf{p}^2+m^2)}$, where the integral is over the hyperboloid $p^2=m^2$ and $p^0\geq 0$, $\tilde{f}$ and $\tilde{g}$ are the Fourier transforms of f and g respectively, and $A^{(-)}$ is the annihilation operator (i.e. negative frequency part) of the field A. Denoting by $R(K)$ the set of all polynomials $P(A(f_1)\cdots)$ with $f_j\in K$, a linear subspace of $\mathcal{D}$, the author derives two properties of K_Ψ: (A) $\Psi\in R(K_\Psi)\Psi_0$; (B) if K is a linear subset of $\mathcal{D}$ such that $\Psi\in R(K)\Psi_0$, then $K\supset K_\Psi$.

{The proof of A as it stands uses the anti-commutation relations rather than the commutation relations, but can be carried through in either case. What the proof of B actually establishes is $K_\Psi\subset K^{\perp\perp}$. (Private communication from the author)} *A. S. Wightman* (Princeton, N.J.)

3005:

Muzikář, Čestmír. Die kovariante phänomenologische Quantentheorie des elektromagnetischen Feldes im Dielektrikum. Czechoslovak. J. Phys. **6** (1956), 409–420. (Russian summary)

The quantized electromagnetic field in the presence of a dielectric is studied. In contradistinction to previous work this author develops the theory covariantly throughout, which he accomplishes by use of the Coulomb gauge instead of the Lorentz gauge. Special emphasis is put on the relation and intercomparison of the Abraham and the Minkowski energy-momentum tensors.

F. Rohrlich (Baltimore, Md.)

3006:

Kalicin, Nikola St. Application of the V. Ritz method in the quantum theory of the field and some generalisations of the Tamm-Dankov equation. Izvestiya Bulgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk Ser. Fiz. **6** (1957), 27–49. (Bulgarian. Russian and English summaries)

The Tamm-Dankoff (T.-D.) equations are derived from a Ritz variational principle in second-quantized formulation, taking free particle occupation numbers as variables. Development of the state vector in two different complete sets of free eigenfunctions then yields equations which involve an arbitrary number of mesons, unlike the T.-D. form, in which each equation only involves three meson amplitudes. A number of examples are treated with the new equations for comparison with T.-D. results.

S. Deser (Waltham, Mass.)

3007:

Nishijima, K. Formulation of field theories of composite particles. Phys. Rev. (2) **111** (1958), 995–1011.

This paper deals with the construction of the S matrix for processes involving composite particles in quantum field theory. The problem is discussed in terms of the Feynman amplitudes $\langle\Omega, T\phi(x_1)\cdots\phi(x_n)\Phi\rangle$ associated with a state $\Phi\rangle$. (Here $\Omega\rangle$ is the physical vacuum and T denotes chronological ordering of the field operators ϕ.) Some of the results are extensions to composite particles of those obtained in an earlier paper [K. Nishijima, Progr. Theoret. Phys. **17** (1957), 765–802; MR **19**, 502].

The author derives integral equations relating the S matrix to the amplitudes of outgoing-wave states $\Phi_\alpha{}^{(+)}\rangle$. By a simple generalization, he obtains recursion formulas between matrix elements of the type $\langle\Phi_\beta{}^{(-)}, T\phi(x_1)\cdots\phi(x_n)\Phi_\alpha{}^{(+)}\rangle$, where $\Phi_\beta{}^{(-)}\rangle$ is an incoming-wave state; these formulas are just the extensions to states containing composite particles of those derived by Lehmann, Symanzik and Zimmermann [Nuovo Cimento (10) **1** (1955), 205–225; MR **17**, 219].

From the T-product recursion formulas, together with invariance under proper inhomogeneous Lorentz transformations and local commutativity, the recursion formulas satisfied by matrix elements of retarded and advanced products are derived [cf. Lehmann, Symanzik and Zimmermann, Nuovo Cimento (10) **6** (1957), 319–333; MR **19**,

1133; Glaser, Lehmann and Zimmermann, Nuovo Cimento (10) 6 (1957), 1122–1128]. Conversely, it is proved that the T-product formulas follow from those for the R- and A-products if it is assumed further that the operator ring formed by $\phi(x)$ at all space-time points is irreducible. The integral equations satisfied by r-functions (vacuum expectation values of R-products) are derived and it is shown that the operator $\phi(x)$ can be reconstructed from a given set of r-functions satisfying these equations and certain boundary conditions. In addition, the recursion formulas for A-products are proved to be equivalent to those for R-products, provided that the CPT theorem holds.

With the help of the T-product formulas, the explicit form of the S matrix is written down. It turns out that, if a composite system built out of elementary ϕ-quanta exists, one may construct a new field ψ from the T-products of ϕ in such a way that in the S matrix the composite system appears as an elementary ψ-quantum. Thus the distinction between elementary and composite particles is largely conventional. [See also W. Zimmermann, Nuovo Cimento (10) **10** (1958), 597–614; R. Haag, Phys. Rev. (2) **112** (1958), 669–673; MR **20**, 6296].

P. W. Higgs (London)

3008:

✱Preuss, H. Integraltafeln zur Quantenchemie. 2ter Bd. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. iv+143 pp. DM 36.00.

This volume, the second in a series on molecular integrals, contains the most comprehensive tables of heteronuclear one-electron integrals yet published. The mesh of the tabulated values and their accuracy (six significant figures) are sufficient to ensure fairly simple and accurate interpolation. An introductory section provides a useful bibliography of other tables and analytic expressions for molecular integrals.

A. C. Hurley (Melbourne)

3009:

✱Bethe, Hans A.; and Salpeter, Edwin E. Quantum mechanics of one- and two-electron atoms. Springer-Verlag, Berlin-Göttingen-Heidelberg; Academic Press Inc., New York, 1957. viii+369 pp. $10.00.

Except for corrections and an addendum this book is identical with an article written in 1956 for the Encyclopedia of Physics [S. Flügge, editor, v. 35, Springer, Berlin, pp. 85–436]. Despite this circumstance, it will undoubtedly be very useful as such. In addition to an "encyclopedic" coverage of the subject of hydrogen-like and helium-like atoms, the authors have striven to present enough of the methodological questions involved explicitly and generally to make their work valuable for a much broader public than the specialists in atomic theory. Another attractive feature is that a number of recent developments in other parts of physics are touched upon so as to make their significance and consequences for atomic theory understandable. Themost prominent case is quantum electrodynamics. After a brief sketch of this part of quantum field theory the authors describe in detail its implications for the calculation of atomic spectra, and they show how, conversely, atomic spectra have provided the best test cases for the validity of our present field-theoretical prescriptions. Another example is a four-page discussion of positronium.

The first chapter covers the properties of one-electron atoms. The non-relativistic and relativistic theories are treated in succession. Radiative corrections are considered extensively, as well as corrections for nuclear motion and structure, fine and hyperfine structures, and positronium. Chapter II deals with helium-like atoms. The first paragraph, covering the non-relativistic theory, discusses the various approximation methods now available for the two-electron system. Then comes the relativistic theory. Considerable attention is paid to the Breit equation for the two-electron problem, and its limits of validity. While chapter III discusses atoms in external fields (Zeeman and Stark effects), the last chapter is concerned with the interaction of atoms with radiation (emission, photo-effect, Bremsstrahlung). Many parts of the discussion are general and thus go beyond the case of one- and two-electron atoms. The book ends with an appendix on spherical harmonics, various addenda and errata, indices for authors, subjects and tables. *L. Van Hove* (Utrecht)

3010:

Pluvinage, P. Approximations systématiques dans la résolution de l'équation de Schrödinger des atomes à deux électrons. I. Principe de la méthode. États *s* **symétriques.** J. Phys. Radium (8) **16** (1955), 675–680.

The author proposes a new method for the solution of the non-relativistic Schrödinger equation for two electrons in a coulomb field (He-like atoms). Let r_1, r_2 be the distances of the electrons from the nucleus and r_{12} their separation. Let $s=r_1+r_2$, $t=r_1-r_2$, $\rho=r_{12}/s$, $\tau=t/s$. The solution of the Schrödinger equation for symmetric S states is written in the form

$$\Psi=\sum_{n=0}^{\infty} F_n(s)Y_n(\rho,\tau),$$

where the F_n satisfy an infinite set of second order ordinary linear differential equations. The F_n are expanded in a series of associated Laguerre polynomials. The coefficients α_{nm} of these expansions satisfy a set of linear equations whose details depend on the choice of the functions $Y_n(\rho,\tau)$. Numerical results are given for HeI. The wave function in fourth approximation yields a slightly worse value for the ground state energy than the Hylleraas three parameter wave function. Reasons are given why this is regarded as satisfactory.

A. S. Wightman (Princeton, N.J.)

3011:

Munschy, G.; et Pluvinage, P. Résolution de l'équation de Schrödinger des atomes à deux électrons. II. Méthode rigoureuse. États *s* **symétriques.** J. Phys. Radium (8) **18** (1957), 157–160.

This paper is a continuation of a previous one by P. Pluvinage (see preceding review). The authors propose a definitive choice of the functions F_{nm} (denoted Y_n in the preceding review) as certain products of Legendre polynomials in τ/ρ, with Jacobi polynomials in ρ. This choice leads to a considerable simplification of the equations for the α_{nm}. *A. S. Wightman* (Princeton, N.J.)

3012:

Munschy, G. Résolution de l'équation de Schrödinger des atomes à deux électrons. III. Suite de la méthode. États *s* **symétriques.** J. Phys. Radium (8) **18** (1957), 552–558.

This paper is a continuation of the two reviewed above. It is devoted to a detailed study of the recurrence relations for the functions F_{nm} and the explicit determination of the coefficients in the equations for the α_{nm}.

A. S. Wightman (Princeton, N.J.)

3013:

Halpern, Francis R. Convergence of the method of moments. Phys. Rev. (2) **111** (1958), 1–2.

A proof is given that the nth approximation of the

method of moments is convergent when applied to the polaron problem. *Author's summary*

3014:

Brulin, O.; and Hjalmars, S. Relativistic wave equations for spin-2 particles with unique mass. Ark. Fys. **14** (1958), 49–60.

Relativistic wave equations of the form $(\beta^k p_k - \lambda)\psi = 0$, where the β^k are square matrices and λ is a multiple of the unit matrix were studied by Bhabha [Rev. Mod. Phys. **17** (1945), 200–216; MR **7**, 272]. With the additional restriction that the commutator $[\beta^k, \beta^l]$ is a multiple of the infinitesimal generator of rotations in the (k, l)-plane, he found that for higher spins ψ described a particle with states of different masses and spins. In the present work, the authors show that particles of a unique rest mass can be obtained if λ is chosen to be a suitable linear combination of invariant matrices. In particular, if ψ is the direct sum of a symmetric second rank tensor and a vector, there are three linearly independent invariant matrices (which are, respectively, the projection operators for the vector, the trace of the tensor and the traceless tensor), from which λ can be obtained as a linear combination. The authors show that there are two possible choices of the coefficients in this linear combination which describe particles of unique mass; only one of these choices leads to positive definite energy. But these particles have a maximum spin 1; the statement that "the corresponding particle can be classified as having spin 2" is misleading, if not incorrect.

E. C. G. Sudarshan (Cambridge, Mass.)

3015:

Hartmann, Hermann. Über ein mechanisches Modell zur Analyse und Darstellung typisch quantentheoretischer Erscheinungen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1957**, 99–103 (1958).

RELATIVITY

See also 3014.

3016:

Gabos, Z.; et Deutsch, R. V. Déduction directe de l'équation Hamilton-Iacobi des équations du type Newton dans la théorie de la relativité restreinte. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. Fiz. **7** (1956), no. 1–4, 79–90. (Romanian. Russian and French summaries)

By adopting several of the principles which the reviewer set as a basis of mechanics (broadening Newton's), the authors deepen the function accomplished by the principle of least action and its relation to Hamilton-Jacobi's equation, concerning the set of possible movements of a material point in a field characterized by a quadridimensional potential (three components for the potential vector and one for the scalar potential).

The corresponding mass is the relativistic one. The development of this paper gives rise to interesting observations. *O. Onicescu* (Bucharest)

3017:

Shibata, Takashi; and Kimura, Toshiei. On the role of Hamiltonians in the relativistic dynamics referred to the new fundamental group of transformations. J. Sci. Hiroshima Univ. Ser. A **21** (1957/58), 15–20.

The author discusses the commutators of spin operators and linear combinations of infinitesimal generators of the Lorentz group. The results are said to be related to the precession of a particle with spin, both in the usual formulation of a relativistic one particle problem and in the formulation based on the author's theory. In the latter theory the role played by the Lorentz group is played by a three-parameter group involving a null-vector.

A. H. Taub (Urbana, Ill.)

3018:

Bohm, David; and Vigier, Jean-Pierre. Relativistic hydrodynamics of rotating fluid masses. Phys. Rev. (2) **109** (1958), 1882–1891.

This paper treats the motion, under special relativity, of a finite mass of fluid ("droplet") subject to no external forces. The centre of mass of the droplet is defined in the usual way, and a new concept, the centre of matter density, is then introduced, first in a special coordinate system, and later for an arbitrary frame. This centre of matter density is then used to define the internal angular momentum of the droplet, and a detailed analysis of the theory of Weysenhoff [Acta Phys. Polon. **9** (1947), 7–18; MR **14**, 213] is given in terms of it.

An extension of the authors' theory, with applications to the theory of elementary particles, is promised in a future paper. *W. B. Bonner* (London)

3019:

Balazs, N. L. On relativistic thermodynamics. Astrophys. J. **128** (1958), 398–405.

This paper deals with the thermodynamic equilibrium of a substance in a static gravitational field. The equilibrium conditions

$$(-g_{44})^{\frac{1}{2}}T=\text{const}; \quad (-g_{44})^{2}P=\text{const}; \quad (-g_{44})^{\frac{1}{2}}\mu=\text{const.},$$

for the distribution of temperature, pressure and chemical potential are obtained. {The reviewer finds it difficult to understand the meaning of the second of these conditions. Also, the author does not seem to be aware that the problem has previously been treated more generally for a mixture of chemical substances in a paper by O. Klein, Rev. Mod. Phys. **21** (1949), 531–533; MR **11**, 468.}

H. A. Buchdahl (Princeton, N.J.)

3020:

Pirani, F. A. E. Invariant formulation of gravitational radiation theory. Phys. Rev. (2) **105** (1957), 1089–1099.

The author classifies Riemannian manifolds obeying the field equations of general relativity locally in an invariant manner, using techniques due to Lichnerowicz [Théories relativistes de la gravitation et de l'électromagnétisme, Masson et Cie., Paris, 1955; MR **17**, 199; p. 33] and to A. Z. Petrov [Kazan Gos. Univ. Uč. Zap. **114** (1954), no. 8, 55–69; MR **17**, 892]. It turns out that there is a (general) type I, representing "no radiation" (more properly perhaps, "impure radiation"), and two (degenerate) types, II and III, of which type II represents pure radiation in a physically intuitive sense, whereas there are no known examples of solutions of type III. To the reviewer's knowledge, this paper represents the first successful attempt to describe invariantly "gravitational waves" within the general theory of relativity.

P. G. Bergmann (New York, N.Y.)

3021:

Tonnelat, Marie-Antoinette. Solution générale des équations $g_{\underset{+}{\mu}\underset{-}{\nu};\varrho}=0$. Expression de la connexion affine en fonction du tenseur fondamental $g_{\mu\nu}$ non dissocié. C. R. Acad. Sci. Paris **246** (1958), 2227–2230.

L'auteur explicite la connexion affine $\Delta_{\mu\nu}{}^{\sigma}$ solution des équations d'Einstein

$$g_{\underset{+}{\mu}\underset{-}{\nu};\varkappa}=\partial_{\rho}g_{\mu\nu}-\Delta_{\mu\rho}{}^{\sigma}g_{\sigma\nu}-\Delta_{\rho\nu}{}^{\sigma}g_{\mu\sigma}=0$$

en fonction du tenseur $g_{\mu\nu}$ non dissocié en ses parties symétriques et antisymétriques par l'intermédiaire de symboles de Christoffel généralisés:

$$[\mu\nu,\rho]=\tfrac{1}{2}(\partial_{\mu}g_{\rho\nu}+\partial_{\nu}g_{\mu\rho}-\partial_{\rho}g_{\mu\nu}).$$

Y. Fourès-Bruhat (Marseille)

3022:

Maurer-Tison, Françoise. Les tenseurs de courbure de deux connexions linéaires associées par l'intermédiaire d'un tenseur régulier de type (0, 2). C. R. Acad. Sci. Paris **246** (1958), 38–40.

Dans une variété différentiable munie d'un champ de tenseurs $g_{\alpha\beta}$ l'auteur a introduit précédemment deux connexions linéaires $\bar{L}_{\beta\gamma}{}^{\alpha}$ et $L_{\beta\gamma}{}^{\alpha}$ [mêmes C. R. **245** (1957), 995–998; MR **19**, 680]. Elle en déduit ici une relation simple entre les formes de courbure associées $\bar{\Omega}=-G\Omega G^{-1}$ où $G(g_{\alpha\beta})$, $\Omega=)(\Omega_{\beta}{}^{\alpha})$. En théorie unitaire du champ, où $\bar{L}_{\beta\gamma}{}^{\alpha}=L_{\gamma\beta}{}^{\alpha}$, cette formule était déjà connue sous le nom de condition d'intégrabilité de Bose-Schrödinger. Elle apparait ici comme propriété géométrique intéressante de l'espace-temps liée a cette condition de champ.

J. Renaudie (Rennes)

ASTRONOMY

See also 2883, 2968, 2985.

3023:

Preisendorfer, Rudolph W. Invariant imbedding relation for the principles of invariance. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 320–323.

In a recent paper [Bellman-Kalaba, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 629–632; MR **18**, 705] the principle of "invariant imbedding" was introduced as an extension of the methodology introduced by Ambarzumian and extended by Chandrasekhar [cf. "Radiative transfer", Oxford Univ. Press, 1950; MR **13**, 136].

In this paper the author applies this principle, which is essentially the observation that physical processes can be described in terms of semi-groups involving other variables apart from time, to transfer problems involving general one-parameter families of surfaces. The foundations for this have been laid in the author's paper, J. Math. Mech. **6** (1957), 685–730 [MR **20**, 5038].

R. Bellman (Santa Monica, Calif.)

3024:

Preisendorfer, Rudolph W. Time-dependent principles of invariance. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 328–332.

The machinery described in the preceding review is now applied to the more interesting and difficult problems arising from time-dependent processes. Relations analogous to those obtained in the stationary case are derived for the time-dependent operators.

R. Bellman (Santa Monica, Calif.)

3025:

Woltjer, L. A theorem on force-free magnetic fields. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 489–491.

The author shows that the total magnetic energy of a suitably defined hydromagnetic system is an extremum when α in Beltrami's equation $\nabla\times\mathbf{H}=\alpha\mathbf{H}$ for force-free magnetic fields is a constant. From this he infers the theorem that "force-free fields with constant α represent the lowest state of magnetic energy which a closed system may attain."

C. H. Papas (Pasadena, Calif.)

3026:

Chandrasekhar, S. On the equilibrium configurations of an incompressible fluid with axisymmetric motions and magnetic fields. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 842–847.

For the case of axisymmetric motions and fields the fundamental equations of the equilibrium states are derived from a variational principle that extremizes the total energy and keeps constant the six integrals of the equation of motion. Of these six integrals three are peculiar to the axisymmetric case and have no analogues in the non-axisymmetric case. It is shown that configurations with axisymmetric magnetic fields and fluid motions which depend on the latter three integrals are not likely to have any practically realizable counterparts.

C. H. Papas (Pasadena, Calif.)

3027:

Oster, Ludwig. Viskosität, elektrische und thermische Leitfähigkeit stellarer Materie. Z. Astrophys. **42** (1957), 228–262.

The theories of electrical and thermal conductivity of a highly ionized gas as developed by Chapman and Cowling (on the basis of two-body Coulomb encounters) and of Spitzer (on the basis of the Fokker-Planck equation with dynamical friction and diffusion) are critically reviewed. Both types of theories involve some assumptions regarding the value of the impact parameter beyond which Coulomb encounters are to be ignored: Chapman and Cowling use the average distance between the ions while Spitzer uses the Debye length. The prevailing opinion is that the latter is correct. Some earlier theories of Sommerfeld [Naturwiss. **15** (1927), 825–832; **16** (1928), 374–381], Persico [Monthly Not. Roy. Astr. Soc. **86** (1926), 93–98] and Gvosdover [Phys. Z. Sowjetunion **12** (1937), 164–181] are also reviewed. The paper includes a tabulation of the relevant coefficients for a variety of physical conditions appropriate for stellar atmospheres.

S. Chandrasekhar (Williams Bay, Wis.)

3028:

Kviz, Zdeněk. On the probability of the discovery of a variable star. Publ. Fac. Sci. Univ. Masaryk **1956**, 193–212. (Russian summary)

The probability of discovery of a variable star by the comparison of pairs of photographic plates of the same area of the sky at different times depends in part on the form of the light curve and on the amplitude. The author presents probability calculations for several forms of light curves, including those typical for RR Lyrae-type variables, cepheids, and long-period variables. The probabilities so obtained disagree in some cases with probabilities obtained by empirical methods. On the other hand, the author remarks, different empirical methods sometimes give substantially different results.

D. Brouwer (New Haven, Conn.)

GEOPHYSICS

See also 2863.

3029:

Simon, R. L. Sur l'existence d'une fréquence critique dans le cas d'une atmosphère illimitée. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 471–476.

The stationary oscillations of an isothermal infinite atmosphere are discussed; the critical frequency found by S. Rosseland is again obtained, but the interpretation, based on the concept of kinetic energy, leads to a somewhat different physical meaning. The case of a Roche atmosphere of infinite radius ($\rho \sim r^{-\alpha}$) is also discussed; in this case, no critical frequency is obtained, owing to the slower decrease of density. *Author's summary*

3030:

Kogan, S. Ya. Application of the method of spherical functions to the problem of scattering of light in the atmosphere. Izv. Akad. Nauk. SSSR. Ser. Geofiz. **1957**, 384–394. (Russian)

3031:

***Кибель, И.А. Введение в гидродинамические методы краткосрочного прогноза погоды. [Kibel', I. A. Introduction to hydrodynamical methods for short-range weather forecasting.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 375 pp. 13.65 rubles.

Table des matières: Introduction; I. Les couches limites atmosphériques; II. Les équations de l'hydrodynamique et thermodynamique de l'atmosphère libre; III. L'ordre de grandeur des éléments météorologiques; IV. Le problème général de la prévision. La tendance du mouvement vers un mouvement géostrophique; V. La prévision pour un niveau moyen. Le problème linéaire; VI. Le problème spatiale de prévision linéarisé; VII. Les équations fondamentales de prévision. Les fonctions de Green pour ces dérivées des éléments météorologiques dans le temps et pour la composante verticale; VIII. Le problème nonlinéaire de prévision pour un niveau moyen. La prévision en utilisant les machines électroniques; IX. Le problème spatiale non linéaire de prévision. La solution à l'aide des machines électroniques; X. Les zones frontales et les surfaces frontales — Mouvements solénoïdaux; XI. Influence de l'orographie (topologie) et le frottement; XII. La transformation calorifique. La prévision dans la stratosphère; Conclusion.

Un travail fondamental des prévisions à courte échéance.

En exposant les théories classiques l'auteur expose en détails certaines méthodes mathématiques et physiques comme par exemple les méthodes de résolution approximatives de certaines équations aux dérivées partielles utilisées par les machines électroniques, certaines théories du rayonnement, etc. . .

Le livre est surtout intéressant par son exposé assez détaillé des théories modernes des savants russes que l'on ne trouve rarement dans les cours météorologiques des pays anglo-saxons et France. *M. Kiveliovitch* (Paris)

3032:

Namikawa, Yoshimasa. On terrestrial geodesic distance. Sûgaku **9** (1957/58), 237. (Japanese)

An approximate formula has been derived by W. D. Lambert [J. Washington Acad. Sci. **32** (1942), 125–130; MR **3**, 300] for the geodesic distance between two given points on the earth regarded as a spheroid. In this note it is shown that this formula must be modified if two points are separated by the vertex of the geodesic through them. *Y. Komatu* (Tokyo)

OPERATIONS RESEARCH AND ECONOMETRICS

See also 2807, 2864, 3039, 3055.

3033:

Aumann, R. J.; and Kruskal, J. B. The coefficients in an allocation problem. Naval Res. Logist. Quart. **5** (1958), 111–123.

The authors suppose that the values of different assignments in an assignment problem are not given objectively but are to be inferred from the answers of a board of experts. The latter are yes-or-no responses to questions about hypothetical assignments of electronic sets to positions on a ship, in their example. The questions are of two types: (1) which of two types of equipment should be installed in a given position; (2) given a current assignment of equipment to positions, which of two possible improvements should be done first? It is shown that the questions imply a set of inequalities on the values of all possible assignments, so that they can be located within a polyhedron which may, in practice, be very small.

In the statement of Theorem 5.2, the words "up to a positive linear transformation" should be inserted after "completely determined." *K. J. Arrow* (Stanford, Calif.)

3034:

***Bowman, Edward H.; and Fetter, Robert B. Analysis for production management.** Richard D. Irwin, Inc., Homewood, Illinois, 1957. xiii+503 pp. $7.00.

This is an elementary textbook, presenting elements of mathematical programming, statistical analysis, maximization procedures, queuing theory, and Monte Carlo methods, with some applications to business management problems. The chapter headings, after three chapters of introduction, are as follows: linear programming; special programming methods; statistical control; sampling inspections; industrial experimentations; total value analysis; incremental analysis; Monte Carlo analysis; and equipment investment analysis. Some applications to inventories, servicing, and lot sizes are included in the chapters on total value and incremental analysis. In the space allotted, no subject can be studied very thoroughly. The book concludes with some case studies.

K. J. Arrow (Stanford, Calif.)

3035:

Kantorovitch, L. On the translocation of masses. Management Sci. **5** (1958), 1–4.

Translation of a Russian paper [Dokl. Akad. Nauk SSSR (N.S.) **37** (1942), 199–201; MR **5**, 174].

3036:

***Saxer, Walter. Versicherungsmathematik. 2ter Teil. Mit einem Anhang von H. Jecklin.** Die Grundlehren der mathematischen Wissenschaften, Bd. 98. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. ix+283 pp. Ladenpreis DM 45.00; Ganzleinen DM 48.60.

This volume completes the author's treatise on actuarial mathematics. While the first volume [MR **17**, 168] is of an elementary character, the present volume deals with advanced topics of actuarial theory. This arrangement permits the author to use analytical tools freely in the second volume. In chapter I actuarial functions are defined as

functions which: (i) are single-valued; (ii) are of bounded variation; (iii) have at most a finite number of discontinuity points; and (iv) are differentiable in all continuity points and have left and right derivatives in the discontinuity points. The author also introduces here upper and lower Cauchy-Stieltjes integrals, as suggested by Schärf [Portugaliae Math. **4** (1944), 73–118; MR **6**, 120]. In chapter II Schärf's method is used to give a unified treatment of the discontinuous and the continuous methods of actuarial mathematics. Chapter III deals with the general risk insurance. This chapter constitutes a remarkable innovation in text books on actuarial mathematics, since its contents apply also to non-life insurance. The first seven sections of this chapter give a very brief survey (mostly without proofs) of some concepts and theorems of probability theory. The emphasis is placed on the discussion of the most important special distributions. In the rest of chapter III (sections 8–12) the author studies the general risk insurance, its premiums, its reserves to protect against random fluctuations, and the related problem of gains and losses. This reviewer regrets the brevity of the sections on the risk process and on risk insurance as well as the omission of the collective theory of risk. While he agrees that the classical theory of risk need not be presented in a modern text book on actuarial mathematics, he hopes that the collective theory of risk will be included in later editions of this book. Chapter IV brings again a useful innovation, not contained in earlier text books, namely a detailed discussion of renewal theory. The discrete as well as the continuous renewal process are discussed, using first a deterministic and then a stochastic approach. Various methods are treated here for the solution of the renewal equation and the asymptotic behaviour of its solutions. Chapter V deals with the graduation of mortality tables. The author discusses first the statistical approach to graduation. From this viewpoint the raw mortality table is a sample from an infinite population and the graduated table is an estimate of the mortality of the population. The customary methods of graduation are then surveyed. Two sections of this chapter deal with properties of orthogonal polynomials and with graduation by means of orthogonal polynomials. One section is devoted to a mechanical method, due to Jecklin and Strickler [Mitt. Verein. Schweiz. Versich. Math. **54** (1954), 125–161; MR **16**, 1134], which reproduces the Makeham formula. Thus, repeated mechanical graduation according to this formula can produce an effect similar to an analytical graduation. It should be remarked that a very general discussion of the use of mechanical formulae for analytical graduations was given in a paper by Walter Gross [Mitt. Öster.-Ungar. Verb. Privat Versich. Anst. (N.S.) **6** (part II) (1911), 89–130], which is too little known. The last section of chapter V deals with methods of testing the goodness of fit of the graduation. The author recommends for this purpose primarily the chi-square test, but also discusses briefly the Kolmogorov-Smirnov test and a sign test.

The book has an appendix of 79 pages by H. Jecklin on the insurance of substandard risks. The first chapter of the appendix deals with basic questions and with the formulation of working hypotheses. Chapter 2 of the appendix treats in an adequate manner the technical methods used in the insurance of substandard risks.

The present volume, as well as the first, testifies to the high standard of actuarial training on the European continent, and will certainly be useful to students and to practical actuaries for a long time.

E. Lukacs (Washington, D.C.)

3037:

✱Gass, Saul I. Linear programming: methods and applications. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. xii+223 pp. $6.50.

The author has succeeded in producing a good elementary text on linear programming. The coverage is adequate, the text is readable, and the exercises are few but good. The numerical examples used to illustrate theory and application are well chosen and in good proportion. The author's experience with this text in his own teaching has obviously helped in making it suitable for use in an introductory course on linear programming.

The book is organized in three parts. Part 1 includes a brief introduction to matrices, vector spaces, convex sets, linear inequalities, and linear equations — as mathematical background for Part 2. Part 2 treats the theoretical and computational aspects of the linear programming problem. The simplex algorithms are discussed in detail, some duality theorems are proved, the problem of solution degeneracy is indicated, and parametric linear programming is introduced. The final chapter in Part 2 presents a useful survey of additional computational techniques and a listing of principal digital-computer codes currently available for solving linear programming problems. Part 3 shows the use of the simplex algorithms in solving the transportation problem, and some of its variations, and closes with a brief but useful survey of other linear-programming applications. The connections with game theory are shown.

This text is entirely suitable for use at the advanced undergraduate level. Except for difficulties in reading references, the mathematical background of the student need not include the calculus. The book also serves well as one for the mature engineer or mathematician to read should he wish to explore modern developments in mathematical programming theory, for he would quickly be led to more advanced topics as they are introduced to him by the author. *M. M. Flood* (Ann Arbor, Mich.)

3038:

✱da Silva Leme, Ruy Aguiar. Aplicação da programação linear ao estudo da decisão dos empresários. [Application of linear programming to the study of business decisions.] Dissertation presented to the Polytechnic School of the University of São Paulo. São Paulo, 1956. vii+172 pp.

Decision theory applied to the problems of the enterprise. Linear programming under certainty and uncertainty. Programming in the long run. There is special emphasis on stochastic linear programming.

G. Tintner (Ames, Iowa)

BIOLOGY AND SOCIOLOGY

3039:

Kochen, Manfred. A mathematical formulation of influence distributions in decision-making groups. J. Soc. Indust. Appl. Math. **6** (1958), 199–208.

In an n-person legislature, let a_{ij} be the probability, assumed the same for all bills, that member i votes for a bill favored only by j. The total influence q_i of i is defined

by $(n-1)q_i=\sum_{j=1}^{n}(a_{ji}-a_{ij})$. Clearly $\sum q_i=0$. Let $\sigma^2=n^{-1}\sum q_i^2$. As a function of the matrix (a_{ij}), σ^2 is maximum if and only if, for some renumbering of the members, (a_{ij}) is triangular with $a_{ij}=1$ for $i<j$. The maximum value is $(n+1)/(3n-3)$. This result may be interpreted as yielding a linear ordering of the legislature, in which each member fully influences all his successors. Several methods of extending these considerations to general bills (favored by more than one member) are discussed.

J. H. Blau (Yellow Springs, Ohio)

3040:

Blank, Albert A. The geometry of vision. British J. Physiol. Opt. **1957**, 1–30.

Approximately two-thirds of the paper is a lengthy discussion of a greatly simplified and superficial model of the ocular apparatus in monocular and binocular vision. An elaborated version of this model is contained in Linksz [Physiology of the eye, Grune and Stratton, New York, 1952; Vol. 2 (Vision)]. The paper concludes with an exposition of some aspects of Luneburg's theory of binocular visual space. {The concept of apparent parallelism as proposed by Luneburg has not been verified experimentally as the author states [see Shipley, J. Opt. Soc. Amer. **47** (1957), 795–803, 804–821]. Hence, it is not true that Blumenfeld's experiment shows that binocular visual space is hyperbolic.}

As the author states, this treatment is not intended for the mathematician.

G. L. Walker (Southbridge, Mass.)

INFORMATION AND COMMUNICATION THEORY

See also 2801, 2802, 2864, 2932.

3041:

Burge, W. H. Sorting, trees, and measures of order. Information and Control **1** (1958), 181–197.

The end points of a tree are assigned positive numbers $v(i)$, $i=1(1)r$, and the value for any other point is taken as the sum of the values of the points of its family. A method is presented for constructing a minimal tree, i.e., a tree for which the sum of the values of all points is a minimum. If the levels are assigned values in sequence, starting with 0 for the root point, if $L(i)$ is the value of the level containing the end point with value $v(i)$, and if c or fewer branches are permitted, then the minimal value is

$$\sum_{i=1}^{r} L(i)v(i)+\sum_{i=1}^{r} v(i)=H_c^r[v(i)]+\sum_{i=1}^{r} v(i).$$

Minimal trees are interpreted as efficient methods for merging sequences of numbers, arranged in strings, into a single sequence, and for constructing minimum redundancy codes once a set of messages and corresponding probabilities are assigned.

Measures of disorder and order in data are defined. Let the data be a permutation of the integers of 1 to n and let s be the class of all permutations made up of r strings with lengths $v(i)$, $i=1(1)r$. The measure of disorder D_s is the minimum number of transfers needed to bring the permutation to complete order by merging:

$$0\leq D_s=H_2^r[v(i)]\leq H_2^n(I),$$

where $H_2^n(I)=\sum_{i=1}^{n}(\log_2 n)+n-1$, and $(\log_2 n)$ denotes the integer next greater than or equal to $\log_2 n$.

C. C. Gotlieb (Toronto, Ont.)

3042:

Wolter, Hans. Zu den Grundtheoremen der Informationstheorie, insbesondere in der Nachrichtentechnik. Arch. Elek. Übertr. **12** (1958), 335–345.

The basic theorems of information theory, viz. the expansion theorem of optics and the sampling theorem of communications, as well as the Küpfmüller-Nyquist relationship, impose as such no ultimate limit to the possible information, either in optics or in communications; such a limit is given, however, by Heisenberg's uncertainty condition of the elementary quantum process and by disturbances. The paper brings into evidence an ambiguity in the proof of the sampling theorem and certain experimental optical inconsistencies concerning the expansion theorem. It is further proven that sharp frequency limits would be contradictory to Maxwell's equations, and even to the causality principle. A rule is stated, according to which a strict conclusion concerning the time function present at the input can be made with any desired accuracy from the time function measured at the end of a communication channel of bandwidth $\Delta\nu$. An electronic computer is described that automatically solves the differential equation governing the problem and converts the bandwidth of a communication channel subsequently into a bandwidth that basically can be enlarged in any desired manner, and its effect is shown by reference to oscillographs. (Author's summary)

S. Kullback (Washington, D.C.)

3043:

Elias, P. Computation in the presence of noise. IBM J. Res. Develop. **2** (1958), 346–353.

The problem of computing reliably with a machine that is itself unreliable has been with us for a long time. This paper attacks the problem by preliminary redundant coding of blocks of information. The paper shows that, in this particular formulation, increasing accuracy requires increasing redundancy so fast that the capacity for computation for a simple combinational computer approaches zero.

R. W. Hamming (Murray Hill, N.J.)

3044:

Freudenthal, Hans. Grundzüge eines Entwurfes einer kosmischen Verkehrssprache. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 352–363.

Lincos is designed to serve as a means of communicating with intelligent creatures living on other cosmic bodies, using radio signals of variable duration and wavelength. The language is to be constructed in such a manner that it is self-explanatory. Therefore, the first subject to be dealt with is mathematics, in which, instead of standard methods of definition and of proof, "quasi-general" definitions and proofs are applied. Then follows the transmission of chronometry, behaviour, and mechanics. {The reviewer wishes to mention a similar, although more embryonic, construction, based on the same principle, by H. Meyer [Synthese **5** (1946), 353–361].}

E. W. Beth (Amsterdam)

CONTROL SYSTEMS

See also 2860, 2861, 2876.

3045:

Emelyanov, S. V. A method for realizing complex control laws using only the error signal or the controlled variable and its derivative. Avtomat. i Telemeh. 18 (1957), 873–885. (Russian. English summary)

The author develops a systematic procedure for realizing nonlinear relationships of the form

$$w(t)=\sigma^*(x, \dot{x})x(t)+\sigma(x, \dot{x})\dot{x}(t),$$

where $\sigma^*(x, \dot{x})$ and $\sigma(x, \dot{x})$ are jump-functions which take on prescribed constant values over specified sectors of the x, $\dot{x}$ plane. Essentially, the method consists in expressing σ^* and σ as sums of "elementary" jump-functions $\Psi_{ij}(x, \dot{x})$ which take on a fixed non-zero value over a sector of the x, $\dot{x}$ plane and vanish elsewhere. The procedure is illustrated by an example involving the use of relays and diodes. *L. A. Zadeh* (New York, N.Y.)

3046:

Rozenvasser, E. N. Stability of nonlinear control systems described by differential equations of the 5th and 6th order. Avtomat. i Telemeh. 19 (1958), 101–113. (Russian. English summary)

On the basis of a theorem of A. I. Lur'e, sufficient conditions of stability "in the large" are obtained for certain control systems described by differential equations of the fifth and sixth order. *H. P. Thielman* (Ames, Iowa)

3047:

Rutkovskiĭ, V. Yu. Analysis of free oscillations of neutral plane without damping of its own and with a relay autopilot. Avtomat. i Telemeh. 19 (1958), 435–447. (Russian. English summary)

A relay system is considered whose linear part is described by the simplest type third-order equation. The analysis of the dynamics of the system is carried out by the method of point transformations of surfaces. The main results of this study are the derivations of equations describing the regions of attraction (on the basis of the initial conditions), the states of equilibrium and a stable limit cycle. These equations can serve as criteria for the selection of the parameters of the regulator for given bounds on the ordinate and on two of its derivatives.
H. P. Thielman (Ames, Iowa)

HISTORY AND BIOGRAPHY

3048:

Mahler, Kurt. On the Chinese remainder theorem. Math. Nachr. 18 (1958), 120–122.

The aim of this expository note is to reproduce the essential mathematical content of the original Chinese method of determining integers x to satisfy the simultaneous congruences, $x\equiv r_i \pmod{m_i}$ $(i=1, 2, \cdots, k)$. In the general case, when the moduli are not necessarily coprime in pairs, this ancient method is quite different from the usual textbook treatment following Gauss.
R. J. Levit (San Francisco, Calif.)

3049:

Hofmann, Joseph E. Zur Geschichte des sogenannten Sechsquadrateproblems. Math. Nachr. 18 (1958), 152–167.

A detailed account of the work of Euler and others on the problem of determining three positive integers such that the sum and the difference of any two is a square.
R. J. Levit (San Francisco, Calif.)

3050:

Freudenthal, Hans. Biographical note on Hermann Weyl. Nederl. Akad. Wetensch. Jboek. (1955/56), 1–8. (Dutch)

3051:

Signorini, Antonio. Commemorazione del Socio Carlo Somigliana. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 21 (1956), 343–351.

3052:

Sansone, Giovanni. Commemorazione del Corrispondente Michele Cipolla. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 21 (1956), 507–523.

Survey of Cipolla's work, by a former colleague. Bibliography.

3053:

Aitken, A. C. The contributions of E. T. Whittaker to algebra and numerical analysis. Proc. Edinburgh Math. Soc. 11 (1958), 31–38.

The article concludes with a brief and just tribute to the Calculus of Observations. Of the original contributions, 8 papers are listed, dealing with the following topics: continued fractions, the solution of algebraic and transcendental equations, determinants having determinants as elements, graduation, and the numerical solution of certain integral equations.
A. S. Householder (Oak Ridge, Tenn.)

3054:

Bott, R.; Hildebrandt, T. H.; Ritt, R. K.; Rothe, E. H.; and Samelson, H. In memoriam Sumner B. Myers: 1910-1955. Michigan Math. J. 5 (1958), 1–4.

A brief biography and a bibliography of 21 entries.

3055:

Kuhn, H. W.; and Tucker, A. W. John von Neumann's work in the theory of games and mathematical economics. Bull. Amer. Math. Soc. 64 (1958), 100–122.

3056:

✱ Мандельштам, Л. И. [Mandel'štam, L. I.] Полное собрание трудов. Том 2. [Complete collected works. Vol. 2.] Edited by S. M. Pytov. Izdat. Akad. Nauk SSSR, 1947. 396 pp. (1 plate) 30 rubles.

The present second volume contains the author's scientific works completed between 1931 and 1944.

GENERAL

3057:

✱ Eves, Howard; and Newsom, Carroll V. An introduction to the foundations and fundamental concepts of mathematics. Rinehart and Company, Inc., New York, 1958. xv+363 pp. $6.75.

This book is a valuable addition to the reference shelf for advanced undergraduate courses of the kind indicated by its title. It has sound scholarship, clear exposition, and appropriate choice of subject matter. Each chapter has an unusually ample number of assorted and often stimulating

exercises, some of which are taken from ancient works. The entire book is strongly historical in flavor. It will make good reading for prospective teachers of mathematics at the secondary level.

The first four chapters are devoted to geometry, largely. They sketch the origins of mathematics in Babylon, Egypt and Greece, the nature and logical defects of Euclid's Elements, non-Euclidean geometry, the axiomatic work of Pasch, Peano, Pieri, and Hilbert, the basic notions of analytic and projective geometry. These chapters are mainly discursive and historical with few proofs in the text, although some theorems are included in the exercises.

Chapter V treats the emergence of the modern abstract notions of algebraic structure. Postulate systems for fields and groups are briefly discussed. Binary operations are so defined that closure is automatic, and therefore closure laws are not found in the text; the alternative, more usual treatment is relegated to the exercises. The role of groups in geometry (Klein's Erlanger Programm) is then taken up. The only serious lapse noted by the reviewer occurs here (p. 137) when a topological transformation of the plane is defined as a transformation $x'=f(x, y)$, $y'=g(x, y)$, where f and g are continuous and single-valued and $\partial(f, g)/\partial(x, y)\neq 0$; there is, of course, no need for derivatives to exist and, if they did, the non-vanishing Jacobian would not suffice to guarantee one-to-one-ness as the standard counterexample $x'=e^x \cos y$, $y'=e^x \sin y$ shows. The chapter closes with a discussion of binary relations and equivalence classes.

Chapter VI expounds the "Modern Mathematical Method" of postulational thinking. Equivalence, consistency, independence, completeness, and categoricalness are discussed.

Chapter VII is devoted to the real number system. It is studied first as a continuously ordered field, and the Archimedean property is derived. Having motivated the deeper study of real numbers by pointing out some of the difficulties in 17th and 18th century analysis, it seems regrettable that the authors do not then go on to give a correct definition of limit and the derivation of a few elementary theorems, including the existence of decimal expressions which are used later in the book. The remainder of the chapter contains an outline of the constructive evolution of the rational, real, and complex number systems, starting with a set of postulates for the natural numbers which is longer and correspondingly easier to develop than Peano's.

Chapter VIII begins with an intuitive introduction to the algebra of sets. A very brief set of postulates, due to Huntington, for Boolean algebra is given, and some theorems are deduced. This is followed by a discussion of transfinite cardinal numbers, and a short introduction to the concept of topological space.

Chapter IX discusses mathematical logic, including the propositional calculus and very brief introductions to the subjects of many-valued logics, the classic antinomies, and the logistic, formalist, and intuitionist points of view.

Three appendices treat constructions with ruler and compasses, Liouville's proof of the existence of transcendental numbers, and the axiom of choice. An extensive bibliography and hints for the solution of some of the exercises conclude the book. *M. Richardson* (Brooklyn, N.Y.)

3058:

✱v. Mangoldt, H. Einführung in die höhere Mathematik. Für Studierende und zum Selbststudium. Seit der sechsten Auflage neu herausgegeben und erweitert von Konrad Knopp. Bd. 2. Differentialrechnung, unendliche Reihen, Elemente der Differentialgeometrie und der Funktionentheorie. 11., verbesserte Auflage. S. Hirzel Verlag, Stuttgart, 1958. xiv+624 pp. DM 23.00.

For remarks applicable to this volume see the review of the first volume (10th edition) in MR **18**, 454.

3059:

✱Pipes, Louis A. Applied mathematics for engineers and physicists. 2nd ed. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. xi+723 pp. $8.75.

The first edition (not reviewed in MR) of this well-known text was published in 1946. In the present edition the chapters on matrix algebra, Fourier methods, variational methods, Laplace transforms, and nonlinear differential equations have been expanded to conform to the greater interest now being shown in the analytical formulation of problems which require computing machines for their numerical solution. There are still no chapters on probability and statistics. A section on Cartesian tensors has been added to the chapter on vector analysis.

3060:

✱Лукомская, А. М. [Lukomskaya, A. M.] Библиографические источники по математике и механике, изданные в СССР за 1917-1952 гг. [Bibliographical source material for mathematics and mechanics published in the USSR in the years 1917-1952.] Under the editorship of V. I. Smirnov. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1957. 354 pp. 11.10 rubles.

A list of 1305 entries, arranged by subject.

3061:

Stone, Marshall H. La matematica e il futuro della scienza. Archimede **10** (1958), 1–16.

Translation of the lecture in Bull. Amer. Math. Soc. **63** (1957), 61–76 [MR **19**, 110.]

BIBLIOGRAPHICAL NOTES

✱Proceedings of the International Congress of Mathematicians, Amsterdam, 1954. Vol. 1. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 582 pp. $7.00.

This volume is the last of the three to appear. It contains organizational data and abstracts of short lectures given at the Congress (these wiil not be reviewed separately); it contains as well some longer lectures, which, like those in Vol. 3, are receiving separate reviews.

Zeitschrift für Angewandte Mathematik und Mechanik **38** (1958), no. 7/8, pp. 253–336.

This issue consists of papers, mostly in summary form, delivered at the Scientific Congress of the Society for Applied Mathematics and Mechanics, April 8–12, 1958, in Saarbrücken, Germany.

Some of the papers are being reviewed separately.

Acta Arithmetica.

Volume 4, no. 1, dated 1958, is a resumption of publication interrupted in 1939. It is published by the Polska

Akademia Nauk, Instytut Matematyczny. Volumes 4 and following may be ordered from Ars Polona, Krakowskie Przedmieście 7, Warsaw; Volumes 1–3 (reprinted) from Johnson Reprint Co., 111 Fifth Ave., New York. All other correspondence should be addressed to Redaction, Acta Arithmetica, ul. Śniadeckich 8, Warsaw 10. The articles are in English, French or German.

Archive for Rational Mechanics and Analysis.
Vol. 1, no. 1 appeared in 1958. Numbers are published irregularly; five numbers comprise a volume. Orders may be placed with Springer-Verlag, Berlin-Göttingen-Heidelberg.

Bulletin Mathématique de la Société des Sciences Mathématiques et Physiques de la République Populaire Roumaine.
Resuming publication interrupted with vol. 48, this journal appears in annual volumes of four numbers each. The new series began with vol. 1(49) 1957. Correspondence should be addressed to Societatea de ştiinţe matematice şi fizice din R.P.R., Redacţia Bulletin Mathématique Bucharest 1, Str. Academiei nr. 14, Romania. Articles are in French, English, Russian, German, Italian, and Romanian.

Canadian Mathematical Bulletin (Bulletin Canadien de Mathématiques).
The official journal of the Canadian Mathematical Congress. Volume 1, no. 1 is dated 1958. The journal contains "information about mathematics and... mathematicians, opinions and conjectures, and original research..., expository writings, a catalogue of events and book reviews". Correspondence may be addressed to the Treasurer, Canadian Mathematical Congress McGill University, Montreal, Canada.

Chiffres.
Publication of the Association française de Calcul, 98 bis, boulevard Arago, Paris XIV^e^. Numbers appear quarterly, in annual volumes, the first of which is dated 1958. Articles are mostly on numerical analysis and tables.

The Computer Journal.
A quarterly journal emphasizing research articles relevant to applications of computers. Vol. 1, no. 1 appeared in 1958. The journal is published by The British Computer Society Limited, Finsbury Court, Finsbury Pavement, London, E.C.2. (The Society also publishes The Computer Bulletin, containing news of interest to its members.)

Funkcialaj Ekvacioj (Serio Internacia).
A semi-annual publication published beginning in 1958, by the Japanese Mathematical Society. Correspondence may be addressed care of Faculty of Science, Kobe University, Milcage-tyo, Higasi-Nada, Kobe, Japan.

Mathematical Tables and Other Aids to Computation.
The new editor of this quarterly, to whom contributed manuscripts should be addressed, is Harry Polachek, David Taylor Model Basin, Washington 7, D.C., U.S.A.

Numerische Mathematik.
Volume 1, no. 1 is dated 1959. The issues will appear irregularly and will be combined into volumes at convenience. Subscriptions may be sent to Springer-Verlag, Berlin-Göttingen-Heidelberg, and other correspondence to any one of 18 listed editors. The articles will chiefly concern numerical and programming methods for digital computers; some attention will also be given to the theory of information.

Pokroky Matematiky, Fysiky a Astronomie.
Volume 1, no. 1 is dated 1957. It is a publication of the Jednota československých matematiků a fysiků (Czechoslovak Mathematical and Physical Society). Orders may be placed with Nakladatelství ČSAV, Vodičkova 40, Praha II. Other correspondence should be addressed to Katedra matematiky a descriptivní geometrie na fakultě elektrotechnické ČVUT v Praze, Na bojišti 3, Praha II. The articles, which include exposition, pedagogy, biography etc., are written chiefly in Czech. The title may be translated Progress in Mathematics, Physics and Astronomy.

Teoriya Veroyatnosteĭ i ee Primeneniya.
Volume 1, no. 1 is dated 1956. It is a publication of the Academy of Sciences of the USSR. Each annual volume contains 4 issues. Articles appear chiefly in Russian, with summaries in English. The title may be translated Theory of Probability and its Applications.

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